On Parameter Estimation and Simulation of Zero Memory Nonlinear Systems

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NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>M, C, K</td>
<td>Mass, Damping and Stiffness matrices</td>
</tr>
<tr>
<td>x</td>
<td>Displacement vector, time domain</td>
</tr>
<tr>
<td>X</td>
<td>Displacement vector, frequency domain</td>
</tr>
<tr>
<td>f</td>
<td>Force vector, time domain</td>
</tr>
<tr>
<td>F</td>
<td>Force vector, frequency domain</td>
</tr>
<tr>
<td>H</td>
<td>Transfer function of underlying linear system</td>
</tr>
<tr>
<td>g(x, ẋ)</td>
<td>Nonlinear function</td>
</tr>
<tr>
<td>P</td>
<td>Nonlinear Coefficient</td>
</tr>
<tr>
<td>R</td>
<td>Input vector to the Reverse Path MISO model</td>
</tr>
<tr>
<td>R</td>
<td>Residue of the underlying linear system</td>
</tr>
<tr>
<td>λ</td>
<td>Pole of the underlying linear system</td>
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<td>B</td>
<td>Transfer function in the Reverse Path MISO model</td>
</tr>
<tr>
<td>A, B</td>
<td>Filter coefficients</td>
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<tr>
<td>ℍ</td>
<td>Fourier transform</td>
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</table>

ABSTRACT

When dealing with nonlinear dynamical systems, it’s important to have well tested, accurate and reliable tools for estimating both the linear and nonlinear system parameters from measured data. The identification technique used in this paper is based on random noise excitation and treats the nonlinear term as a feedback force acting on an underlying linear system. The parameter estimation is done in the frequency domain using conventional MIMO/MISO techniques. Two different formulations are studied in this paper, the first one is known as “reverse-path” and the second one is called “Nonlinear Identification through Feedback of Outputs”. Both these formulations are based on a technique initially developed by Julius S. Bendat.

Both these methods are applied to a continuous mechanical system, a beam structure with a geometrical nonlinearity. The different results obtained are then compared to each other. Also, a very fast way to simulate forced response of the estimated nonlinear system, based on residues and poles and the use of digital filters, is demonstrated.
1. INTRODUCTION

In the past years much effort has been put into the field of parameter estimation of nonlinear dynamical systems. A large number of methods have been developed, but there is still no method which is applicable to a general structure with an arbitrary nonlinearity. An important method, initially developed by Julius S. Bendat [1], treats the nonlinearity as a force feedback term acting on an underlying linear system. The parameter estimation is done in the frequency domain and is known as the Reverse Path (RP). A method based on the same principles as RP is NIFO, Nonlinear Identification Through Feedback of Outputs, proposed by Adams and Allemang [2]. NIFO has an advantage over RP because it can easily be extended to multiple degree of freedom systems with several nonlinearities. While on the other hand RP is less sensitive to force dropouts and disturbances in the force signal that might occur around the resonances.

A frequently reoccurring discussion connected to nonlinear analysis with random data, especially when using the Reverse Path method, is the need for conditioned inputs. This has led to the formulation of the CRP method, Conditioned Reverse Path [1, 3]. However, the use of conditioned inputs complicates the problem formulation and increases computational cost considerably. Also, for all systems tested in both simulations and in measurements, no improvement has been obtained by using conditioned instead of unconditioned inputs.

The aim of parameter identification/estimation is to find suitable parameters to a mathematical model of a system. Once these parameters are identified, the model may be used to predict the behavior of the system. Since simulation of nonlinear systems can be a very time consuming and difficult task, an innovative and fast method developed by the authors will also be briefly described in this paper.

2. RANDOM NOISE EXCITATION OF NONLINEAR SYSTEMS

In this chapter a general approach to estimate the nonlinear coefficients of a system with an arbitrary amount of zero memory nonlinearities using the unconditioned Reverse Path method is presented. This approach assumes that the location of the nonlinearities is known beforehand and that it’s possible to measure the responses in the nonlinear nodes. However, the driving point doesn’t have to be located on a nonlinear node. For a deeper discussion about locating nonlinear nodes, see [4]. Also in the derivation below a single excitation force is assumed i.e. single input.

A nonlinear MDOF system can in time domain be expressed by the following equation:

\[
M\ddot{x} + C\dot{x} + Kx + nl(x, \dot{x}) = f
\]  

(1)

Where \( nl \) is the nonlinear restoring forces acting on the system, these forces can be dependent on either \( x \) or \( \dot{x} \). The nonlinear restoring forces are expressed more clearly in equation 2.

\[
nl = \sum_{m}^{N_{el}} P_{m} \cdot w_{m} \cdot g_{m}(x, \dot{x})
\]  

(2)

Where \( P_{m} \) is the coefficient related to the nonlinear function \( g_{m} \) and \( w_{m} \) is a position vector describing where the nonlinearity is located. So, every nonlinear element is described by a coefficient \( P_{m} \), a nonlinear function \( g_{m} \) and a position vector \( w_{m} \). Examples of nonlinear functions are given in equation 3 and 4.

\[
g_{m} = w_{m}^{T} \cdot x \left| w_{m}^{T} \cdot x \right| \]  

(Square hardening spring)  

\[
g_{m} = (w_{m}^{T} \cdot \dot{x})^2 \]  

(Quadratic damping)
Combining equation 1 and 2 in the frequency domain gives the following expression:

\[
X = H \left( F - \sum_{m} P_m \cdot w_m \cdot \mathcal{Z}(g_m(x, \dot{x})) \right)
\]

(5)

This expression is the same as used in the derivation of NIFO, where the Force and nonlinear feedback forces are used as input, see [5]. In RP the displacements and the nonlinear feedback forces are used as input, and the force as output. As mentioned above it is assumed that the system is only excited in one point, this means that only one row in the transfer matrix \( H \) is obtained. Therefore it’s not possible to invert the matrix in order to obtain the equation needed to directly formulate the RP method. However, by formulating an input vector \( R \) to be used as input to the MISO model shown in figure 2.1, a transfer function \( B \) can be calculated.

![Figure 2.1](image_url)

**Figure 2.1 – Displacement and nonlinear restoring forces are used as input and the force as output in a MISO model.**

If we apply the \( H_1 \) estimate to this system the transfer function \( B \) is obtained by equation 6, where \( N \) is the number of measured responses:

\[
B^i = G_{R^i F} \cdot G_{R^i R}^{-1} \quad i = [1 \quad 2 \ldots \quad N]
\]

(6)

The vectors \( R^i \) are given by:

\[
R^i = [X_i^r \quad r_1 \quad r_2 \ldots \quad r_N^r]^T
\]

(7)

Where \( r_m \) is the Fourier transform of the nonlinear function \( g_m \), according to equation 8

\[
r_m = \mathcal{Z}(g_m(x, \dot{x}))
\]

(8)

As seen above the vector \( R^i \) contains the displacement of degree of freedom \( i \) in frequency domain and the Fourier transforms of the nonlinear functions. As also evident the only difference between the vectors \( R^i \) and \( R^{i+1} \) is the linear displacement \( X_i^r \), everything else remains the same. Therefore it’s possible to calculate as many transfer functions \( B^i \) as there are measured responses.

When all the transfer functions \( B^i \) are calculated the nonlinear coefficients \( P_m \) can be obtained according to equation 9

\[
P_m = \frac{1}{B_{inp}^i} \cdot \sum_{i=1}^{N} w_m(i) \cdot \frac{1}{B_{inp}^i} \cdot \mathcal{Z}(g_m(x, \dot{x}))
\]

(9)

Where the superscript refers to the \( B \) vector used, the subscript tells the specific position in that vector and \( w_m(i) \) refers to the value at the position \( i \) in the vector \( w_m \). \( inp \) means Input i.e. the driving point. Since \( B_{inp}^i \) is known
for all measured DOFs, it’s possible to calculate the corresponding transfer functions of the underlying linear system $H$ according to equation 10.

$$H_{inp}^{i} \frac{1}{B_{inp}^{i}} \quad \text{(10)}$$

The method described above can also be used to estimate polynomial nonlinearities, as for example structural play or bilinear stiffness. It can easily be extended to several input forces, but as long as forces are not applied in every measurement point the simplest way is still to set up the system as a MISO system. The benefit of applying extra forces, except it might be easier to excite the nonlinearity, is that for every transfer function $B$ calculated, it’s possible to estimate as many transfer functions of the underlying linear system as there are driving points. This fact will off course affect the simulation time needed for the parameter estimation in a positive way.

By applying the method above, all the transfer functions of the underlying linear system in the driving point column are obtained, as well as estimates of all the nonlinear coefficients $P_{mn}$. By modal parameter estimation it’s therefore possible to obtain the complete transfer function of the underlying linear system.

3 TIME RESPONSE SIMULATION OF NONLINEAR MECHANICAL SYSTEMS

There exist several methods to calculate the time response of a mechanical system to an arbitrary excitation. A few of these methods are: the convolution integral, the state transition matrix method and the Runge-Kutta method with variations. A disadvantage with the convolution integral is the computational cost due to its non-recursive nature. However the integral can be used to derive a recursive algorithm, a digital filter, which reduces the computational cost significantly compared with the convolution integral.

3.1 LINEAR SYSTEMS

Working with discrete sampled data demands a transformation from the continuous time domain into the discrete time domain. The basis for a family of methods is the convolution integral:

$$x(t) = \int_{0}^{t} h(t-\tau)f(\tau)d\tau \quad \text{(11)}$$

The convolution integral can be used with any linear time invariant system. According to the modal superposition theorem a frequency response function can be expressed using partial fraction expansion of the residues $R_{r}$ and poles $\lambda_{r}$ as follows:

$$H(s) = \sum_{i=1}^{N} \frac{R_{r}}{s - \lambda_{r}} + \frac{R_{r}^{*}}{s - \lambda_{r}^{*}} \quad \text{(12)}$$

In equation 12, $s$ is the Laplace variable and $N$ denotes the total number of modes included. To simplify the upcoming derivation of the digital filter, only one mode, $r$, will be considered. The complex conjugate in the equation above is also removed in order to simplify the expressions further. In the derivation, a temporary function $H^{D}$ will be used.

$$H^{D}(s) = \frac{R_{r}}{s - \lambda_{r}} \quad \text{(13)}$$
Equation 13 has the impulse response:

\[ h^d(t) = R_e e^{\lambda t} \]  

(14)

Changing to discrete time, the response \( x_r(nT + T) \), where \( T \) is sampling interval, can be calculated using the convolution integral from equation 11 with \( x \) as output and \( f \) as input:

\[
x_r(nT + T) = \int_{0}^{nT+T} R_r e^{\lambda (nT+T-\tau)} \cdot f(\tau) d\tau
\]

\[
= R_r e^{\lambda T} \cdot x_r(nT) + R_r e^{\lambda T} \int_{0}^{T} e^{-\lambda u} f(u + nT) du
\]

(15)

From the equation above, it's obvious that \( x_r(nT + T) \) can be calculated with a recursion formula using only \( x_r(nT) \) and the input signal in the interval \([f(nT) \cdot f(nT + T)]\). The way the samples \( f(t) \) are treated in the evaluation of the last integral defines the design method, i.e. Impulse invariant, Step invariant or Ramp invariant.

As an example, the step invariant method uses the constant value \( f(nT) \) in the entire interval. Therefore equation 15 becomes:

\[
x_r(nT + T) = R_r e^{\lambda T} \cdot x_r(nT) + f(nT) \frac{R_r}{\lambda_r} (e^{\lambda T} - 1)
\]

(16)

Equation 16 can be expressed in the Z-domain:

\[
H^D(z) = \frac{z^{-1} \frac{R_r}{\lambda_r} (e^{\lambda T} - 1)}{1 - z^{-1} R_r e^{\lambda T}}
\]

(17)

Adding the complex conjugate and considering \( N \) number of modes, the transfer function \( H \) can be expressed according to equation 18 below:

\[
H(z) = \sum_{r=1}^{N} \left[ \frac{z^{-1} \frac{R_r}{\lambda_r} (e^{\lambda T} - 1)}{1 - z^{-1} R_r e^{\lambda T}} + \frac{z^{-1} \frac{R_r^*}{\lambda_r^*} (e^{\lambda T} - 1)}{1 - z^{-1} R_r^* e^{\lambda T}} \right]
\]

(18)

Equation 18 shows that one set of filter coefficients \( A \) and \( B \) is needed for each individual mode included in the simulation. It might seem complicated to calculate these filter coefficients, but by putting the numerator and denominator in vector form according to the specified time delays and then use convolution, the filter coefficients are obtained. In equation 19 below, a formula to calculate the corresponding filter coefficients to equation 18 is shown. \( N \) stands for numerator and \( D \) for denominator:
These filter coefficients are obtained using step invariant, other methods can also be used and then the change occurs in the numerator and depends on how the sample in the last integral in equation 15 is treated. The denominator stays the same regardless of which method is used. The time response can be calculated according to the difference equation below:

$$x(n) = \sum_{r=1}^{N} B_r^0 f(n) + B_r^1 f(n-1) + \ldots + B_r^m f(n-m) - A_r^1 x_r(n-1) - \ldots - A_r^n x_r(n-p)$$

As seen in equation 20, one digital filter is obtained for every mode included. Using the superposition theorem, the total response $x(n)$ is the sum of the individual digital filters.

### 3.2 Extending to Nonlinear Systems

The equation of motion for a system in discrete time domain with a single displacement dependent nonlinearity can be expressed in z-domain, as:

$$Z(z) \cdot X(z) + NL(z) = F(z)$$

Where $Z(z)$ is the impedance matrix of the underlying linear system and $NL(z)$ is the nonlinear function. The system response can be expressed as:

$$X(z) = H(z) \cdot (F(z) - NL(z))$$

The nonlinearity is simply considered as an extra force acting on the underlying linear system. Therefore the response of the nonlinear system can be expressed with a few small modifications to the difference equation 20 used for linear systems. The difference equation for a system with a single displacement dependent nonlinearity connected to ground is shown below:

$$x(n) = \sum_{r=1}^{N} B_r^0 (f(n) - nl(n)) + B_r^1 (f(n-1) - nl(n-1)) + \ldots + B_r^m (f(n-m) - nl(n-m)) - A_r^1 x_r(n-1) - \ldots - A_r^n x_r(n-p)$$

The only unknown parameters in equation 23 are the displacement $x(n)$ and the nonlinear force $f(n) - nl(n)$, since $nl(n)$ is a function of $x(n)$ and $f(n)$ is a known value, a nonlinear equation is obtained which needs to be solved for every single mode at each time step. This can be done with any suitable numerical method. Once these equations are solved modal superposition can be adopted and the response from each individual digital filter sums up to the total response, exactly as in the linear case.
This method can be extended to nonlinear systems with an arbitrary amount of nonlinearities connected between any degrees of freedom. The algorithm can be made very fast, so time simulations of systems with several degrees of freedom can be done in a very efficient way. Also, a huge benefit of using residues and poles as base in the algorithm is that they can be obtained either from MCK-systems, experimental measurements, analytical models or FE-models which of course gives a very wide application area. For a detailed description of the errors involved in these simulations see [6].

4 SIMULATIONS

The system used consists of a cantilever beam with two cubic nonlinearities, see figure 4.1. One nonlinearity is connected between DOF a and ground and the second one between DOF b and c. The force is applied in DOF d, away from the nonlinearities. In the simulation the first five modes of the beam are taken into account. To further explain the method presented in chapter two, a few steps in the process will be shown by specific equations, only valid for this model, which are related back to the corresponding general equation in chapter two.

![Figure 4.1 – The cantilever beam with two nonlinearities used in the simulation. The mass of the beam is 1.1 kg and the first resonance frequency is 10Hz, the first five modes were considered. 50% overlap and a hanning window were used.](image)

The system in figure 4.1 can be expressed in frequency domain according to equation 24 below. Where $H$ is the transfer function of the underlying linear system and all nonlinearities are expressed as external forces acting on that system. For this analysis, the input vectors $w_1$ and $w_2$ from chapter two becomes, $[1 \ 0 \ 0 \ 0]$ and $[0 \ 1 \ -1 \ 0]$ respectively.

$$
\begin{bmatrix}
H_{aa} & H_{ab} & H_{ac} & H_{ad} \\
H_{ba} & H_{bb} & H_{bc} & H_{bd} \\
H_{ca} & H_{cb} & H_{cc} & H_{cd} \\
H_{da} & H_{db} & H_{dc} & H_{dd}
\end{bmatrix}
\begin{bmatrix}
-P_1 \cdot \Im(x_a^3) \\
-P_2 \cdot \Im((x_b - x_c)^3) \\
P_2 \cdot \Im((x_b - x_c)^3) \\
F
\end{bmatrix}
= \begin{bmatrix}
X_a \\
X_b \\
X_c \\
X_d
\end{bmatrix}
$$

From this equation, an expression for the displacement $X_d$ can be obtained as shown in equation 25.

$$F \cdot H_{dd} - P_1 \cdot (H_{da}) \cdot \Im(x_a^3) - P_2 \cdot (H_{db} - H_{dc}) \cdot \Im((x_b - x_c)^3) = X_d$$

Solving for $F$ and putting the equation in matrix form yields the following expression:
Equation 26 contains the transfer function \( B_d \), from equation 6 in chapter two. From this equation it’s possible to get an estimate of \( H_{dd} \) of the underlying linear system. By calculating the rest of the transfer functions \( B^e \), \( B^h \) and \( B^a \), all unknowns in equation 26 can be obtained and the nonlinear coefficients solved for by using equation 9 from chapter two.

The settings and nonlinear coefficients used in the simulation are all listed in table 4.1. The simulation was carried out in MATLAB [7] on a laptop with an Intel Pentium M processor of 1.7GHz and 512 Mb RAM, the total simulation time was 330 seconds.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal Length</td>
<td>( 2^{21} ) samples</td>
</tr>
<tr>
<td>Blocksize</td>
<td>( 2^{17} ) samples</td>
</tr>
<tr>
<td>Fs</td>
<td>4096 Hz</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>7e6</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>3e8</td>
</tr>
</tbody>
</table>

Table 4.1 – Settings and coefficients used in the simulation

The frequency response functions of the underlying linear system for all responses were calculated as well as the two nonlinear coefficients using the method derived in chapter two. The estimated linear transfer function from DOF a were then compared with the true transfer function of the underlying linear system. The results are shown in figure 4.2 to 4.4.

Figure 4.2 clearly shows that the estimated underlying linear system is very close to the true linear system. Also a huge improvement between the FRF obtained if the system isn’t considered nonlinear, the Raw-FRF and the FRF obtained from the nonlinear analysis can be seen.
The multiple coherence is often used as a measure of the goodness-of-fit for the estimated nonlinear system. As seen in figure 4.3, the multiple coherence is almost equal to unity for all frequencies, indicating a very good fit. Also a big improvement compared with the raw-coherence obtained by linear analysis can be seen.

Figure 4.4 shows that the estimated nonlinear coefficients are constant over the entire frequency range, as expected. If the studied system contains zero memory nonlinearities, the estimated coefficients should be real after averaging. Therefore only the real part of the estimated coefficients is used in the analysis. The estimated coefficients differs maximum 2% from the theoretical value.

5. EXPERIMENTAL TEST

The test rig chosen is a T-beam with a cubic nonlinearity at one end. This beam has two closely spaced modes, the first bending and torsional mode, both highly affected by the nonlinearity. The nonlinearity is obtained by using two slender beams fixed in the end points. These beams will, when subjected to large displacements, behave nonlinear due to the effect of geometrical nonlinearity. The system can be seen as a linear system with a local nonlinearity connected to ground.
The system is excited in point 5 and responses are measured in points 1 to 6, where point 5 is the nonlinear node. Two rigid tables are used in the experimental setup in order to reduce the risk of bias errors as much as possible. A picture and a schematic drawing of the test rig are shown in figure 5.1 below:

![Figure 5.1](image)

**Figure 5.1** – The experimental test rig, a T-beam with a geometrical nonlinearity. One end of the T-beam and both ends of the two slender beams are firmly fixed to rigid tables. Measurements are given in [mm].

Since the nonlinearity affects the system more at high vibration levels, it’s important to use a sufficiently high excitation force so that a clear nonlinear effect is visible. The RMS value of the force used in this experimental test is $14.3 \, N$. Due to force dropouts around the resonances, the nonlinear effects are reduced substantially if a flat force spectrum is supplied to the shaker. The force dropouts need to be compensated for by adding extra energy into the force signal around the resonances in order to excite the nonlinearity in a sufficient way for the nonlinear analysis. As mentioned earlier, two nonlinear identification techniques were tested on the experimental data, Reverse Path and NIFO. None of these methods demands that the system is excited in the nonlinear node in order to work. However, due to physical limitations and the need to excite the torsional mode, node 4, 3 and 6 couldn’t be used. When the system was excited in node 1 and 2, the weak nature of the beam made it very hard to excite the nonlinearity enough for successful parameter estimation.

All force signals were created in MATLAB and Signal Calc Mobilyzer [8] was used to collect the time data. The data analysis was carried out in MATLAB.

Also, a static measurement was performed in order to get a second estimate of the nonlinearity to compare with the results from the dynamic experiments.

### 5.1 Analysis of Experimental Data

There are a few uncertainties when performing nonlinear analysis on experimentally acquired data, for example the integration from acceleration to displacement and the polynomial form of the nonlinearity. It’s not for certain the exponent of the nonlinearity is an even number, see for example [9], and the parameters used in the integration process can change the final result significantly. As mentioned previously, the multiple coherence function is a measure of the goodness-of-fit of the estimated nonlinear system. Therefore, the multiple coherence function can be used as an error function in order to find the most suitable parameters for the integration and also the best estimate of the nonlinear power. In this case the inverse of the area under the multiple coherence function was used as error estimate and the built in MATLAB function `fminsearch` was used to optimize the integration and to find the best estimate of the nonlinear exponent in the studied system. The best fit was obtained with $|x|^{3.5872}\cdot \text{sgn}(x)$ as nonlinear function. This is the nonlinear function used in the analysis that follows. All calculations are done with a sampling frequency of 2604.2 Hz, a block size of 2*15 samples, 50% overlap and 120 averages.

Figure 5.2 shows the raw coherence from linear analysis (SISO) compared with the multiple coherence obtained from the nonlinear analysis (MISO). The nonlinear effect is clearly visible in the raw-coherence function, especially
around the resonances. There is also a clear disturbance in the coherence at approximately three times the resonance frequencies, indicating a close to cubic nonlinearity. Also, a huge improvement in the coherence can be seen after the nonlinear analysis, indicating a good estimate of the nonlinearity. The multiple coherence function from RP is slightly better than the one obtained using NIFO.

![Figure 5.2 – Raw coherence compared with multiple coherence obtained by Reverse-Path and NIFO respectively.](image)

In figure 5.3 the estimated FRFs of the underlying linear system obtained from the nonlinear analysis is compared to the Raw-FRF obtained by linear analysis. Again, the nonlinear effect is evident in the heavily distorted raw FRF. Also evident is the huge difference in amplitude around the resonances in the estimates of the underlying linear system obtained by RP and NIFO. This has been observed before in for example [10, 11]. One possible reason for this is disturbances, uncorrelated noise, present in the force signal around the resonances. Since RP minimizes the effect of noise on the force channel, the estimate of the underlying linear system becomes better with RP compared to NIFO.

![Figure 5.3 – Transfer functions obtained from the nonlinear analysis using RP and NIFO compared with the transfer function obtained using linear analysis, Raw - FRF.](image)

In figure 5.4 the absolute value of the estimated nonlinear coefficients are plotted. The estimation is more or less constant with exception for a small disturbance in the region between $[60, 70] \text{Hz}$. By looking at the Raw coherence in figure 5.2, it is clear that the coherence is 1 above $55 \text{Hz}$ with exception for the region around $90 \text{Hz}$. This indicates a perfect linear relationship between the input and output of the system. Thus, there is no
nonlinear effect at these frequencies. Therefore, the frequency span between [1.5 55] Hz was used to estimate the nonlinear coefficient since this is the frequency span where the nonlinear effect is clear. Also, only the real part of the estimated coefficient was used. The real part is approximately 10 times bigger than the imaginary part, for both methods.

\[ |x^{2.5827} \cdot \text{sgn}(x)| \]

Figure 5.4 – Nonlinear coefficients obtained by Reverse-Path and NIFO

To get an extra assurance of the validity of the estimated parameters from RP and NIFO, a static measurement was performed on the structure. The nonlinear coefficient obtained from this measurement is compared to the nonlinear coefficient from the RP analysis in figure 5.5 below. The difference between the estimated nonlinear coefficients using RP and NIFO and the static measurement is between 3-4%. All the estimated nonlinear coefficients are listed in table 5.1.

\[ |x^{2.5827} \cdot \text{sgn}(x)| \]

Figure 5.5 – Static measured forces vs. displacement and the estimated function from the static measurement in the left figure. The nonlinear function estimated from the static measurement compared to the nonlinear function estimated by Reverse Path to the right.

<table>
<thead>
<tr>
<th>Nonlinear Coefficient</th>
<th>Reverse-Path (1.2224 \cdot 10^8)</th>
<th>NIFO (1.2376 \cdot 10^8)</th>
<th>Static Measurement (1.2740 \cdot 10^8)</th>
</tr>
</thead>
</table>

Table 5.1 – The estimated nonlinear coefficients
As a final test of the validity of the estimated system, a time response simulation was carried out using the input force from the measurements. The forced response of node 2 was simulated and compared to the measured response of node 2. The result from the RP analysis was used to calculate the residues and poles of the underlying linear system. Also, the estimated nonlinearity from RP was used in the simulation. The results are shown in figure 5.6.

![Figure 5.6](image)

**Figure 5.6** – Measured time response in node 2 compared with the simulated time response of node 2 on the left hand side. To the right the PSDs of the two responses are compared to each other.

As seen to the left in figure 5.6 the simulated and measured response amplitudes are very similar. In fact the difference in standard deviation between the two responses is only 3%. To get an estimate of the frequency content in the signals, the PSDs of the measured and simulated response are compared to the right in figure 5.6. The match between the PSDs is also very good, especially at the high displacement levels.

### 5. CONCLUSIONS

In chapter two a general approach for analyzing MDOF systems with several nonlinearities and only one input force using the Reverse Path method was developed. This method is based on using partially correlated inputs which makes the formulation much simpler than the well known Conditioned Reverse Path method (CRP). This approach was tested both in a simulation and on an experimental test rig with good results.

Two methods, based on random noise excitation, Reverse Path and NIFO was used in the experimental test. Both these methods treats the nonlinearity as a force feedback term making it possible to estimate both the underlying linear system as well as the nonlinear coefficients, and thereby get sufficient information to make a mathematical model of the system from one single measurement. Both methods used have been proven to work with experimental data. The difference between the estimated nonlinear coefficients using RP and NIFO is very small. However, the estimation of the underlying linear system becomes poor when using NIFO on experimental data and can, in this case, not be used to get a good estimate of the residues and poles.

A mathematical model was created from the data obtained by the RP analysis and used to simulate a forced response. The model could predict the standard deviation of the displacement of the system within 3%. When comparing the PSDs of the simulated and measured response, a close match was observed, indicating a very good agreement between the model and the tested structure.

### 6. REFERENCES


[8] SignalCalc Mobilyzer v 4.0.000, Data Physics Corporation, 1997-2002

