Estimation of Impulse Responses: a novel Method and its Use in Experimental Modal Analysis

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Abstract

A novel method for the estimation of Impulse Response matrices from measured input-output data, called IRE (Impulse Response Estimation) is presented. It is shown that the method is strongly consistent under colored output noise. Its performance is compared to two other methods achieving the same goal. Then, the IRE algorithm is combined with classical impulse response driven algorithms for experimental modal analysis (pLSCE, ERA and ITD) and with a new algorithm, IREAR (Impulse Response Estimation And Realization). A simulation example shows that IREAR outperforms the three classical algorithms.

Nomenclature

\[ 
\begin{align*}
\dagger & \quad \text{Moore-Penrose pseudo-inverse} \\
\ast & \quad \text{deterministic, noiseless part} \\
\ast & \quad \text{stochastic, noisy part} \\
c_{1:2} & \quad \text{columns } c_1 \text{ to } c_2 \\
r_{1:2} & \quad \text{rows } r_1 \text{ to } r_2 \\
\hat{\mu}(\bullet) & \quad \text{sample mean} \\
\hat{\sigma}(\bullet) & \quad \text{sample standard deviation} \\
\sigma(\bullet) & \quad \text{standard deviation} \\
E[\bullet] & \quad \text{expectation operator} 
\end{align*} 
\]

1 Introduction

The estimation of a (finite part) of the impulse response from measured data is an important problem in sound and vibration engineering. For instance, many Multiple Degree Of Freedom (MDOF) time domain methods for Experimental Modal Analysis (EMA) that are very often used, such as the poly-reference Least Squares Complex Exponential method (pLSCE) [4, 13], the Ibrahim Time Domain method (ITD) [3, 4, 13], and the Eigensystem Realization Algorithm (ERA) [4, 13], need an estimate of the impulse response matrices (also called Markov parameters [7]) to start from. Hence, the determination of a non-parametric estimate of the impulse response is crucial for these EMA techniques, since the quality of the modal parameters obtained from the methods described above depends heavily on the initial impulse response estimates.

An obvious method for the estimation of the impulse response is the use of direct measurements (i.e. measuring the system’s response to an impulse input), but its main weakness is that many physical systems do not allow impulse inputs of such an amplitude that the signal to noise ratio (SNR) of the measured outputs is high enough for them to be accurate [12]. Söderström and Stoica [19] presented a simple and unbiased method to estimate impulse responses from input-output data that is valid only if the applied inputs are noise-free and have a white noise nature. The usual practice for the estimation of impulse responses from data with arbitrary input signals is therefore to calculate the (discrete) inverse Fourier transform of a frequency response function estimate [7]. As an alternative, Phan et al. [16] and Juang [7] present the Observer/Kalman filter IDentification (OKID) for the estimation of impulse responses. This algorithm is not statistically consistent (meaning that, for an infinite number of data samples, it does not yield the exact estimates with probability one [17]).
However, recent research in the area of subspace identification by Markovsky et al. [14] yielded a new and efficient algorithm for the exact computation of a finite part of the impulse response directly from noiseless input-output data using arbitrary input signals (the only condition that lies upon the inputs is persistency of excitation, a very weak condition on which we will return later). In this paper, the algorithm is modified such that it is statistically consistent with respect to output noise that is uncorrelated with the applied forces. It follows that if classical EMA techniques are combined with the proposed Impulse Response Estimation (IRE) algorithm, they are statistically consistent as well.

Although there exist other EMA techniques that are also consistent, substantial improvement of pLSCE, ITD and ERA is important since

1. These methods are widely used in modal testing, as evidenced by their appearance in the main textbooks on the subject [3, 4, 13];
2. These methods are less complicated than other methods that are exact for noiseless data and consistent for noisy output data. Hence, they are preferable when their statistical performance is comparable.

It should be noted that another desirable statistical property, namely statistical efficiency (meaning that, for a finite number of data samples, the variance of the obtained estimates is minimum, i.e. reaches the Cramér-Rao lower bound), is not guaranteed for any non-iterative EMA algorithm. Therefore, both the bias errors and the variance errors of estimates obtained from pLSCE, ITD and ERA in this paper are compared in a numerical example.

Finally, a new EMA technique called IREAR (Impulse Response Estimation and Realization) is proposed and compared with the other EMA algorithms in the numerical experiment. From this experiment, it follows that IREAR produces the most accurate modal parameter estimates, i.e. the bias and variance errors are the lowest with respect to the other methods.

2 Computation of impulse response matrices

2.1 A Consistent algorithm starting from input-output data

The algorithm presented here is a generalized version of the algorithm for noiseless data presented in [14], for the determination of the first \( N_m \) impulse response matrices \( H_k, k = 0, \ldots, N_m - 1 \) of a Linear Time Invariant (LTI) system from measured inputs and outputs. The algorithm is named IRE (Impulse Response Estimation).

Denote by \( H_{0:N_m-1} \) the matrix consisting of the considered impulse response matrices, i.e.,

\[
H_{0:N_m-1} = \begin{bmatrix} H_0^T & H_1^T & \cdots & H_{N_m-1}^T \end{bmatrix}^T
\]  

(1)

First, define a block Hankel matrix of measured outputs \( y_k \in \mathbb{R}^1, k \in \{0, \ldots, N-1\} \), corrupted by zero-mean ergodic and wide-sense stationary noise \( y^*_k \):

\[
y_k = y^*_k + \nu_k \quad Y_{k_1|k_2} = \begin{bmatrix} y^{k_1^1} & y^{k_1+1^1} & \cdots & y^{k_2^1} \\ \vdots & \vdots & \ddots & \vdots \\ y^{k_1^3} & y^{k_1+1^3} & \cdots & y^{k_2^3} \end{bmatrix}^T \quad Y^{k_3} = \begin{bmatrix} y_{k_3} & y_{k_3+1} & \cdots & y_{k_3+j-1} \end{bmatrix}
\]

where \( j \) is determined by \( j = N - N_m - \ell + 1 \) and \( \ell \) is strictly larger than the system’s time lag [21], which for linear mechanical systems equals the system order. Thus, an upper bound on the system order \( \ell \) is assumed to be known: \( \ell > n \).

Define also a block Hankel matrix of measured zero-mean inputs \( u_k \in \mathbb{R}^m, k \in \{0, \ldots, N-1\} \), corrupted by zero-mean ergodic and wide-sense stationary noise \( u^*_k \):

\[
u_k = u^*_k + u_k \quad U_{k_1|k_2} = \begin{bmatrix} U^{k_1^1} & U^{k_1+1^1} & \cdots & U^{k_2^1} \\ \vdots & \vdots & \ddots & \vdots \\ U^{k_1^3} & U^{k_1+1^3} & \cdots & U^{k_2^3} \end{bmatrix}^T \quad U^{k_3} = \begin{bmatrix} u_{k_3} & u_{k_3+1} & \cdots & u_{k_3+j-1} \end{bmatrix}
\]

Construct the matrix \( D \in \mathbb{R}^{(m+l)(\ell+N_m)\times j} \) as

\[
D = \begin{bmatrix} W_1 U_1^T U_1 & W_2 U_2^T U_2 \end{bmatrix}
\]

\[
W_1 = W_{0|N_m+\ell-1}^{1:j} \quad W_2 = W_{0|N_m+\ell-1}^{1:j} \quad W_{0|N_m+\ell-1} = \begin{bmatrix} U_{0|N_m+\ell-1}^T & Y_{0|N_m+\ell-1}^T \end{bmatrix}^T
\]

\[
U_1 = U_{0|N_m+\ell-1}^{1:j} \quad U_2 = U_{0|N_m+\ell-1}^{1:j}
\]

where \( \bullet^T \) denotes the transposes and \( \bullet^\dagger \) denotes the Moore-Penrose pseudo-inverse [1]. Note that \( U_1^T U_1 \) and \( U_2^T U_2 \) are the orthogonal projectors onto the row spaces of \( U_1 \) and \( U_2 \), respectively [20]. Finally, divide \( D \) into the following 2 blocks:

\[
D = \begin{bmatrix} D_a \ D_b \end{bmatrix} = \begin{bmatrix} D_{1:(N_m+\ell)m+l} \ D_{(N_m+\ell)m+l}(N_m+\ell)(m+l) \end{bmatrix}
\]

where \( \bullet_{r_1:r_2} \) denotes the rows \( r_1 \) to \( r_2 \).

With the following assumptions:
we have that the following estimate for the FRF \( H \) and Juang [7].

From fig. 1, it can be seen that the system has two closely spaced modes.

The following implementation is computationally efficient and robust.

Algorithm 1 (Consistent Impulse Response Estimation (IRE)).

1. Compose \( W_1 \) and \( W_2 \)
2. Compute \( L_{11}, L_{21}, Q_1 \) and \( L_{31}, L_{41}, Q_3 \) from the following LQ factorizations where \( L \) is lower triangular, \( Q \) is orthonormal, \( L_{11}, L_{31} \in \mathbb{R}^{(N_m+\ell)\times (N_m+\ell)}, \) \( Q_1 \in \mathbb{R}^{(N_m+\ell)\times \ell}, \) \( Q_3 \in \mathbb{R}^{(N_m+\ell)(N_m+\ell)} \): \[
W_1 = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} Q_1 = \begin{bmatrix} L_{31} \\ L_{41} \end{bmatrix} \]
3. Compute \( D \) from \[
D = \begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix} Q_1 = \begin{bmatrix} L_{31} \\ L_{41} \end{bmatrix} Q_3
\]
4. Compute \( L_{51} \in \mathbb{R}^{(N_m+\ell)\times N_m}, L_{61} \) from the LQ factorization of \( D \):
\[
D = LQ = \begin{bmatrix} L_{51} \\ L_{61} \end{bmatrix} Q
\]
5. Compute \( \hat{H}_{0|N_m-1} = L_{61} L_{51}^\top \) \( O \) where \( O \) is given by (4).

2.3 Example

In this section, the performance of the IRE algorithm is compared with the performance of two other algorithms: (i) taking the inverse Fourier transform of the \( H \) estimate for the FRF [4] and (ii) the OKID algorithm of Phan et al. [16] and Juang [7].

This is done by means of a numerical experiment. The system under test is a 2DOF mechanical system, where the inputs are forces and the outputs accelerations, described by the following state space equations:

\[ x_{k+1} = Ax_k + Bu_k \]
\[ y_k^d = Cx_k + Du_k \]

where \( x_k \in \mathbb{R}^n \) is the state of the system, \( y_k \in \mathbb{R}^d \) the vector with measured outputs and \( u_k \in \mathbb{R}^m \) the vector with measured inputs. In this example, the system matrices are

\[
A = \begin{bmatrix} 0.9 & 0.5 & 0 & 0 \\ -0.6 & 0.7 & 0 & 0 \\ 0 & 0.4 & -0.7 & 0 \\ 0 & 0.6 & 0.9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0.9 \\ 0.4 & 0.7 \\ 0.6 & 0.5 \\ 0.8 & 0.3 \end{bmatrix},
\]
\[
C = 10^{-3} \begin{bmatrix} 0.3 & 0.5 & 0.7 & 0.9 \\ 0.8 & 0.6 & 0.4 & 0.2 \\ 0.5 & 0.4 & 0.1 & 0.6 \end{bmatrix}, \quad D = 10^{-3} \begin{bmatrix} 1.8 & 1.7 \\ 1.6 & 1.5 \\ 0.6 & 1.2 \end{bmatrix}
\]

From fig. 1, it can be seen that the system has two closely spaced modes.
Figure 1: Amplitude and angle of element (1, 1) of the frequency response function corresponding to system (5-7) as a function of frequency divided by the sampling frequency.

Figure 2: Simulation 1: sample bias error (a) and sample standard deviation (b) of element (1, 1) of the estimated impulse response matrix \( \hat{H}_k \). Full line: IRE, dotted line: OKID, dashed line: inverse \( H_1 \).

The forces applied as inputs consist of random excitation that is uniformly distributed in the interval \([-0.5N, 0.5N]\). It is assumed that the measured outputs are corrupted by noise caused by the vibration of the same structure due to ambient white noise Gaussian excitation. So the noiseless outputs \( y^d_k \) are corrupted by additive colored measurement noise \( y^s_k \), generated by

\[
\begin{align*}
x_{k+1,y} &= A x_{k,y} + B f^s_k \\
y^s_k &= C x_{k,y} + D f^s_k
\end{align*}
\]

where \( f^s_k \) is white Gaussian noise with standard deviation \( \sigma(f^s_k) = 0.05 \). As a result, the SNR of the outputs is between 4dB and 11dB.

With this system description, a sequence of 1000 Monte Carlo runs is performed. In each run, another realization of the unmeasured input \( f^s_k \) is used for the generation of 1000 output samples \( y_k = y^d_k + y^s_k \). The following parameter choices are made: \( N = 1000, N_m = 10 \) and \( \ell = 10 \).

The sample bias error \( \vert H_k - \hat{\mu}(\hat{H}_k) \vert \), where \( \hat{\mu}(\bullet) \) represents the sample mean, estimated over all 1000 Monte Carlo runs, is shown in fig. 2a. It can be noticed that IRE and OKID yield smaller bias errors than inverse \( H_1 \) estimation. The sample standard deviation error \( \hat{\sigma}(\hat{H}_k(1,1)) \), estimated over all 1000 Monte Carlo runs, is shown in fig. 2b. While the bias error is the most important error for inverse \( H_1 \) estimation, the variance error is dominant for IRE and OKID. The variance error of the OKID estimates is slightly lower than the variance error of the IRE estimates.

In a second series of 1000 Monte Carlo simulations, the unmeasured input \( f^s_k \) is simulated starting from the noiseless signal of fig. 3a, which shows a dip near the resonance of the system (5-6). To this signal, white Gaussian random noise with standard deviation \( \sigma = 0.5 \) is added in the frequency domain. A resulting realization is shown in fig. 3b.
Fig. 3: Simulation 2: Unmeasured input: noiseless signal (a) and a realization after adding Gaussian random noise (b).

Fig. 4: Simulation 2: sample bias error (a) and sample standard deviation (b) of element (1, 1) of the estimated impulse response matrix $\hat{H}_k$. Full line: IRE, dotted line: OKID, dashed line: inverse $H_1$.

Fig. 4a shows the sample bias error and fig. 4b shows the sample standard deviation, estimated over 1000 Monte Carlo simulations. It can be seen that in this case, the bias error is significantly lower for IRE, while the variance error is similar for all methods. In fact, for OKID the bias and variance errors are of the same magnitude. Hence the mean squared error is lowest for IRE.

It can be concluded that IRE is an accurate method for nonparametric impulse response estimation that clearly outperforms inverse $H_1$ estimation. In addition, it has better theoretical statistical properties than OKID since it is a strongly consistent algorithm. The simulation example indicates that the performance of IRE is comparable to that of OKID, or sometimes even better.

3 The poly-reference Least Squares Complex Exponential (pLSCE) algorithm

The purpose of this section is to provide a brief overview of the poly-reference implementation of the LSCE algorithm. More details and references can be found in [4, 13].

The impulse response matrices $H_k$ can be decomposed into modal parameters:

$$H_k = \sum_{l=1}^{n} \varphi_l l_t^T e^{\lambda_{e,l} \Delta t_k} + \sum_{l=1}^{n} \varphi_l l_t^T \lambda_{l}^k = \Phi \Lambda^{k-1} L^T, \quad k \geq 1$$

where $\varphi_l$ is the mode shape of mode $l$, $l_t$ is the corresponding modal participation vector, $\lambda_{e,l}$ and $\lambda_{l}$ are the
corresponding continuous-time and discrete-time poles, $\Delta t$ is the discrete time step and

$$\Phi = [\varphi_1 \ldots \varphi_n], \quad A = \begin{bmatrix} \lambda_1 & 0 \\ & \ddots & \ddots \\ 0 & & \lambda_n \end{bmatrix} \quad \text{and} \quad L = [l_1 \ldots l_n]$$

The estimation of the poles $\lambda_i$ and modal participation vectors $l_i$ follows a procedure originally developed by Prony [18] for SISO systems and expanded later for MIMO systems [13].

The algorithm starts by grouping the impulse response matrices in a block Toeplitz matrix and solving the following set of equations in a least squares sense:

$$H = \begin{bmatrix} H_1 & H_{1-2} & \cdots & H_i \\ H_{i+1} & H_i & \cdots & H_{i-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{2i-1} & H_{2i-3} & \cdots & H_i \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{i-1} \end{bmatrix} = 0, \quad A_k \in \mathbb{C}^{m \times m}$$

If the constraint $A_{i-1} = I$ is used during the solution, it follows from [10, 9, 2] that the mathematical modes that arise when the model order $n$ is overestimated, are unstable, i.e. they have negative damping.

From (10), it follows that if $m_i \geq n$, any row of $H^{1|1}$ can be written as a linear combination of the following set of linearly independent row vectors

$$[l_i^T \lambda_i^{i-2} \ldots l_i^T], \quad i = 1, \ldots, n$$

Since $A_f$ belongs to the right null space of $H^{1|1}$, it follows that

$$l_i^T A_0 \lambda_i^{i-1} + l_i^T A_1 \lambda_i^{i-2} + \cdots + l_i^T A_{i-1} = 0, \quad i = 1, \ldots, n$$

From this equation, it follows that $\lambda_i$ and $l_i$ can be obtained from the following eigenvalue decomposition:

$$\begin{bmatrix} -A_0^{-T} A_1^T & -A_0^{-T} A_2^T & \cdots & -A_0^{-T} A_{i-1}^T \\ I_{(i-1)m} & & & \end{bmatrix} \begin{bmatrix} \lambda_i^{i-2} l_i \\ \lambda_i^{i-3} l_i \\ \vdots \\ l_i \end{bmatrix} = \lambda_i \begin{bmatrix} \lambda_i^{i-2} l_i \\ \lambda_i^{i-3} l_i \\ \vdots \\ l_i \end{bmatrix}$$

The mode shapes need to be estimated in a second step. Since in (10), $\Phi$ is the only unknown at this point, it can be estimated in a least squares sense from [15]:

$$\Phi = [H_1 \ H_{i-1} \ \ldots \ \ H_i] (\Lambda^{i-1} L^T \ \Lambda^{i-2} L^T \ \ldots \ L^T)^T$$

### 4 The Impulse Response Estimation And Realization (IREAR) algorithm and the Eigensystem Realization Algorithm (ERA)

#### 4.1 IREAR

The realization algorithms considered in this paper are used for the identification of the system matrices $(A, B, C, D)$ of the deterministic state-space model (5-6). With this model, it is straightforward to write the impulse response matrices $H_k$ as a product of system matrices:

$$H_0 = D, \quad H_i = CA^{i-1}B, \ i \geq 1 \quad (11)$$

From this equation, the direct transmission term $D$ can be obtained directly. The realization starts with gathering the other Markov parameters in a block Hankel matrix [5]:

$$H_{3|1} = \begin{bmatrix} H_1 & H_2 & \cdots & H_i \\ H_2 & H_3 & \cdots & H_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_i & H_{i+1} & \cdots & H_{2i-1} \end{bmatrix} \quad (12)$$
where \( i \) is chosen in such a way that, if \( n \) is the expected system order, \( li \geq n, mi \geq n \) and \( i \geq 2 \). The block Hankel matrix decomposes into the extended observability matrix \( \mathcal{O}_i \) and the extended controllability matrix \( \mathcal{C}_i \) [5]:

\[
H_{1|i} = \begin{bmatrix} \mathcal{C} & \mathcal{C}A & \cdots & \mathcal{C}A^{i-1} \\ 0 & \mathcal{C} & \cdots & \mathcal{C}A^{i-2} \end{bmatrix} [B \ AB \ \cdots \ A^{i-1}B] = \mathcal{O}_i \mathcal{C}_i
\]

(13)

The matrices \( \mathcal{O}_i \) and \( \mathcal{C}_i \) can be obtained from \( H_{1|i} \), up to a similarity transformation of the \( \mathcal{A} \) matrix, using reduced singular value decomposition [22, 11]:

\[
H_{1|i} = USV^T \quad \mathcal{O}_i = US^{1/2} \quad \mathcal{C}_i = S^{1/2}V^T
\]

(14)

where \( S \in \mathbb{R}^{n \times n} \) contains only the nonzero singular values and \( U \in \mathbb{R}^{n \times n} \) and \( V \in \mathbb{R}^{m \times n} \) contain the corresponding singular vectors. If the Markov parameters \( \mathcal{H}_l \) are exact, the number of nonzero singular values equals the system order. If not, the system order is lower than the number of nonzero singular values; it then needs to be estimated as the number of dominant singular values, and in this case \( S \) contains only the dominant singular values and \( U \) and \( V \) contain the corresponding singular vectors [22, 11].

The \( \mathcal{C} \) matrix can be determined as the first \( l \) rows of \( \mathcal{O}_i \), and the \( \mathcal{B} \) matrix can be determined as the first \( m \) columns of \( \mathcal{C}_i \). For the determination of \( \mathcal{A} \), one can use of the shift structure of the matrix \( \mathcal{O}_i \) [11]:

\[
\mathcal{A} = \mathcal{O}_i | \mathcal{C}_i
\]

(15)

where \( \mathcal{O}_i \) is equal to \( \mathcal{O}_i \) without the last \( l \) rows and \( \mathcal{C}_i \) is equal to \( \mathcal{O}_i \) without the first \( l \) rows.

In this paper, the combination of the IRE algorithm of section 2 and Kung’s realization algorithm [11], described in this section, is named IREAR (Impulse Response Estimation And Realization).

### 4.2 ERA

Juang and Pappa [8, 7] proposed to use the realization algorithm developed by Zeiger and McEwin [22] for Experimental Modal Analysis, and to name it Eigensystem Realization algorithm (ERA). It follows exactly the same lines as Kung’s algorithm discussed in the previous section. The only difference is that, for the estimation of \( \mathcal{A} \), the structure of the block Hankel matrix is exploited:

\[
H_{2|i+1} = \mathcal{O}_i \mathcal{A} \mathcal{C}_i \quad \mathcal{A} = \mathcal{O}_i \mathcal{H}_{2|i+1} \mathcal{C}_i^\dagger
\]

In [7], Juang presents an extension of the ERA algorithm called Eigensystem Realization Algorithm with Data Correlations (ERA/DO). It follows the main lines of ERA, except for the fact that correlations of Markov parameters are used as primary data instead of the Markov parameters themselves.

### 5 The Ibrahim Time Domain (ITD) algorithm

The Ibrahim time domain method [6] can be regarded as a system realization method where the system matrices are identified directly in a modal basis. Therefore, the factorization (13) reads

\[
H_{1|i} = \begin{bmatrix} \Phi & \Phi \Lambda & \cdots & \Phi \Lambda^{i-1} \end{bmatrix} [L^T \ \Lambda L^T \ \cdots \ \Lambda^{i-1}L^T] = \mathcal{O}_i \mathcal{C}_i^m
\]

and if follows immediately that

\[
H_{2|i+1} = \mathcal{O}_i \mathcal{A} \mathcal{C}_i^m
\]

Since \( \mathcal{O}_i^m \mathcal{C}_i^m = I \) and \( \mathcal{C}_i^m \mathcal{C}_i^m \mathcal{C}_i^m = I \), we have that

\[
H_{2|i+1} = \mathcal{O}_i \mathcal{A} \mathcal{C}_i^m = \mathcal{O}_i \mathcal{A} \mathcal{O}_i^m \mathcal{C}_i^m = \mathcal{O}_i \mathcal{A} \mathcal{O}_i \mathcal{C}_i^m = \mathcal{O}_i \mathcal{A} \mathcal{O}_i \mathcal{C}_i^m \mathcal{C}_i^m = \mathcal{H}_{1|i} \mathcal{C}_i^m \mathcal{A} \mathcal{C}_i^m
\]

so the reduced eigenvalue decomposition of \( \mathcal{V} \) yields \( \Phi \) and \( \Lambda \) and the reduced eigenvalue decomposition of \( \mathcal{W} \) yields \( L \) and \( \Lambda \).
6 Example

The purpose of this section is to compare the performance of the pLSCE, IREAR, ERA and ITD algorithms in a numerical experiment. For the calculation of the impulse responses, the IRE algorithm as presented in section 2 is used, where $N_m = 10$ and $\ell = 10$.

The system (5-7) is used again together with the noise model (8-9). A series of 1000 Monte Carlo simulations is performed, where the measured and unmeasured forces are the same as in the first set of Monte Carlo simulations of section 2.3. In each Monte Carlo run, the modal parameters are calculated with pLSCE, IREAR, ERA and ITD for a system order of 4. The sampling frequency is 100Hz.

The sample bias and the sample standard deviation of the undamped eigenfrequencies and damping ratios are shown in table 1 and 2, respectively. The sample mean of the MAC between exact and estimated mode shapes, and the sample standard deviations of these MAC values, are shown in table 3.

From these tables, it can be concluded that IREAR shows the lowest bias and variance errors. The bias errors of ERA are as low as the bias of IREAR, but the variance errors are slightly higher. The bias and variance errors of ITD and pLSCE are considerably higher than the errors of IREAR and ERA.

<table>
<thead>
<tr>
<th>$f_{udi}$ [Hz]</th>
<th>$f_{udi}$ pLSCE bias</th>
<th>$f_{udi}$ IREAR bias</th>
<th>$f_{udi}$ ERA bias</th>
<th>$f_{udi}$ ITD bias</th>
</tr>
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<tbody>
<tr>
<td>9.447</td>
<td>0.038</td>
<td>0.005</td>
<td>0.007</td>
<td>0.004</td>
</tr>
<tr>
<td>12.000</td>
<td>0.17</td>
<td>0.128</td>
<td>0.133</td>
<td>0.199</td>
</tr>
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</table>

Table 1: Numerical experiment: sample bias and sample standard deviation $\hat{\sigma}$ of the undamped eigenfrequencies $f_{udi}$, estimated using 1000 Monte Carlo simulations.

<table>
<thead>
<tr>
<th>$\xi_i$ [%]</th>
<th>$\xi_i$ pLSCE bias</th>
<th>$\xi_i$ IREAR bias</th>
<th>$\xi_i$ ERA bias</th>
<th>$\xi_i$ ITD bias</th>
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</thead>
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<tr>
<td>6.113</td>
<td>0.038</td>
<td>0.005</td>
<td>0.007</td>
<td>0.004</td>
</tr>
<tr>
<td>16.476</td>
<td>0.17</td>
<td>0.128</td>
<td>0.133</td>
<td>0.200</td>
</tr>
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</table>

Table 2: Numerical experiment: sample bias and sample standard deviation $\hat{\sigma}$ of the damping ratios $\xi_i$, estimated using 1000 Monte Carlo simulations.

<table>
<thead>
<tr>
<th>$f_{udi}$ [Hz]</th>
<th>MAC pLSCE $\hat{\mu}$</th>
<th>MAC pLSCE $\hat{\sigma}$</th>
<th>MAC IREAR $\hat{\mu}$</th>
<th>MAC IREAR $\hat{\sigma}$</th>
<th>MAC ERA $\hat{\mu}$</th>
<th>MAC ERA $\hat{\sigma}$</th>
<th>MAC ITD $\hat{\mu}$</th>
<th>MAC ITD $\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.447</td>
<td>0.9964</td>
<td>0.0064</td>
<td>0.9994</td>
<td>0.0007</td>
<td>0.9992</td>
<td>0.0009</td>
<td>0.9984</td>
<td>0.0019</td>
</tr>
<tr>
<td>12.000</td>
<td>0.9974</td>
<td>0.0115</td>
<td>0.9989</td>
<td>0.0015</td>
<td>0.9987</td>
<td>0.0018</td>
<td>0.9979</td>
<td>0.0028</td>
</tr>
</tbody>
</table>

Table 3: Numerical experiment: sample mean $\hat{\mu}$ and sample standard deviation $\hat{\sigma}$ of the MAC values between the real and estimated mode shapes, estimated using 1000 Monte Carlo simulations.

7 Conclusions

First, a novel method for the estimation of impulse responses from input-output data, called IRE (Impulse Response Estimation), was presented and compared to two other methods: inverse FRF ($H_1$) estimation and OKID. It was shown that the performance of IRE and OKID is much better than inverse $H_1$ estimation. IRE has theoretically better statistical properties than OKID (it is strongly consistent under colored output noise), which makes it most probably more robust. This was confirmed by a numerical example.

Then, a new technique for experimental modal analysis called IREAR was presented and its performance was compared to the performance of three impulse response driven EMA techniques (pLSCE, ERA and ITD) using a numerical example. For the impulse responses that are used as inputs in these algorithms, IRE was used, so that all algorithms are strongly consistent under colored output noise. It can be concluded that

1. IREAR yields the best estimates for the modal parameters (lowest bias and variance errors);
2. ERA shows bias errors that are as low as for IREAR, but the variance errors are clearly higher;
3. ITD shows considerably higher bias and variance errors than IREAR and ERA.

4. although pLSCE also shows considerably higher bias and variance errors than IREAR and ERA, it might be preferable in situations where the stabilization diagram is not clear, since a clear discrimination between physical and mathematical poles can be made based on the sign of the damping ratio (mathematical poles have negative damping).

A Proof of the main theorem

Proof. First consider the noiseless case \( \forall k : y_k^0 = 0 \). Under the assumptions of the theorem one has the following exact relationship [14]:

\[
W_{0|N_m+\ell-1} \tilde{G} = [O_{\mathcal{H}_{0|N_m-1}}]
\]

where \( \tilde{G} \) can be solved exactly from the first \((N_m + \ell)m + \ell \) rows, so that \( \mathcal{H}_{0|N_m-1} \) can be calculated exactly from the last \( N_m \ell \) rows.

A trajectory of length \( L \) of a discrete linear time invariant system is defined as a sequence of \( L \) input-output combinations: \( \bar{W}_L = \{(u_k, y_k), \ldots, (u_{k+L-1}, y_{k+L-1})\} \). Under the assumptions of the theorem any valid trajectory \( \bar{W}_{N_m+\ell} \) can be obtained from a linear combination of the columns of \( W_{0|N_m+\ell-1} \) [21]. This means that the columns of \( [O^T \mathcal{H}_{0|N_m-1}]^T \) are valid trajectories of length \( N_m + \ell \).

Using (5-6), one has the decomposition

\[
Y_{0|N_m+\ell-1} = O_{N_m} X_0 + \mathcal{F}_{N_m} U_{0|N_m+\ell-1}
\]

where

\[
X_0 = \begin{bmatrix} x_0 & x_1 & \cdots & x_j \end{bmatrix} \quad O_{N_m} = \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^{N_m-1})^T \end{bmatrix}^T
\]

\[
\mathcal{F}_{N_m} = \begin{bmatrix} H_0 & 0 & 0 & \cdots & 0 \\ H_1 & H_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_{N_m-1} & H_{N_m-2} & \cdots & H_{N_m-3} & H_0 \end{bmatrix}
\]

Since all valid trajectories of length \( N_m + \ell \) can be obtained from linear combinations of the columns of \( W_{0|N_m+\ell-1} \), all initial conditions have to be allowed and \( \text{rank}(X_0) = n \). From the persistency of excitation condition, it follows that \( \text{rank}(U_{0|N_m+\ell-1}) = (N_m + \ell)m \). Using the definition \( r_0 = \text{rank}(O_{N_m}) \), it follows from (16) that

\[
\text{rank}(W_{0|N_m+\ell-1}) = \min(r_0, n) + (N_m + \ell)m
\]

With the following definitions

\[
Y_1 = Y_{0|N_m+\ell-1}^{1:fl(j/2)} \quad Y_2 = Y_{0|N_m+\ell-1}^{fl(j/2)+1:j} \quad X_1 = X_0^{1:fl(j/2)} \quad X_2 = X_0^{fl(j/2)+1:j}
\]

the decomposition (16) is written as

\[
\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = O_{N_m} \begin{bmatrix} X_1 & X_2 \end{bmatrix} + \mathcal{F}_{N_m} U_{0|N_m+\ell-1}
\]

from which one has

\[
\begin{bmatrix} Y_1 U_1 & Y_2 U_2 \end{bmatrix} = O_{N_m} X_1 U_1^T(U_1 U_1^T)^{-1} U_1 + O_{N_m} X_2 U_2^T(U_2 U_2^T)^{-1} U_2 + \mathcal{F}_{N_m} U_{0|N_m+\ell-1}
\]

From the persistency of excitation condition, \( U_1 \) and \( U_2 \) are of full row rank, and as a consequence

\[
\text{rowspace}(U_1) \cap \text{rowspace}(U_2) = \emptyset
\]

\[
\Downarrow
\]

\[
\text{rowspace}(U_1) \cap \text{rowspace}(U_2) = \emptyset
\]

where \( \emptyset \) is the empty set. Since additionally

\[
\text{rowspace}(U_2) \subset \{\text{rowspace}(U_1) \cup \text{rowspace}(U_2)\}
\]

and given that \( [Y_1 U_1] [Y_2 U_2] \) equals the last \((N_m + \ell)\) rows of \( D \), one has that

\[
\text{rank}(D) = \min(r_0, n) (N_m + \ell) + (N_m + \ell)m = \text{rank}(W_{0|N_m+\ell-1})
\]
where the last equality follows from (17) and the fact that, by definition, \( \ell > n \). Using this result and, since \( D \) consists of linear combinations of columns of \( W_0|N_{m + \ell - 1} \),

\[
\text{colspan}(D) \subseteq \text{colspan}(W_0|N_{m + \ell - 1})
\]

one has

\[
\text{colspan}(D) = \text{colspan}(W_0|N_{m + \ell - 1})
\]

As a result, there exists a matrix \( G \) such that

\[
DG = \begin{bmatrix} \mathcal{O} \\ \mathcal{H}_0|N_{m - 1} \end{bmatrix}
\]

where \( G \) can be solved exactly from the first \( (N_m + \ell)m + \ell^2 \) rows, so that \( \mathcal{H}_0|N_{m - 1} \) can be calculated exactly from the last \( N_m \ell \) rows.

Consider now the case where \( y_k \) is corrupted by noise that is independent from the inputs. One has [17]

\[
\text{a.s.} \lim_{j \to \infty} \mathcal{H}_0|N_{m - 1} = \left( \text{a.s.} \lim_{j \to \infty} \frac{D_aD_a^T}{j} \right)^{-1} \mathcal{O}
\]

so only the strong consistency of \( D \) remains to be shown. This follows from

\[
\text{a.s.} \lim_{j \to \infty} Y_1U_1^T = \lim_{j \to \infty} \frac{Y_1^dU_1^T}{j} \quad \text{a.s.} \lim_{j \to \infty} Y_2U_2^T = \lim_{j \to \infty} \frac{Y_2^dU_2^T}{j}
\]

where \( \cdot^d \) denotes the noiseless part. \( \square \)

References


