Nonlinear Dynamics of a Post-buckled Beam: A Parametric Space Investigation

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ABSTRACT

The nonlinear response of an aircraft panel due to acoustic and thermal loading is computationally expensive using traditional finite element methods. This has lead to the use of several reduced-order modal model development methods for such systems. In this paper, we first provide a historical review of the nonlinear phenomenon associated with the post-buckling response due to thermal-acoustic loading with specific emphasis on snap-through response. Next, a parametric space investigation is presented to study the effect of thermal gradient and/or preload on the post-buckling dynamic behavior of a clamped-clamped beam system. This paper studies a single mode approximation of the clamped-clamped beam. These reduced order models are based on the implicit condensation method. First the parametric space investigation is conducted to identify the stability of the periodic solutions for both models. The effect of harmonic excitation is presented in terms of the bifurcation diagram with specific focus on the effect of initial pre-load parameter. Finally, some conclusions about the response to multi-frequency excitation are presented in terms of the numerical bifurcation diagrams.

INTRODUCTION

Future Air Force vehicles will require a significantly expanded operating envelope. Aircraft structural panels will be subjected not only to aerodynamic loading (mechanical loading), but also to aerodynamic heating (thermal loading). These structural panels are attached to relatively cooler substructures (i.e., spars and ribs, both of which act as heat sinks); the panels are, therefore, constrained from free expansion. These constraints will cause thermal stresses to build up in the panels. High-intensity thermal loading could cause thermal buckling, thermal yielding and thermal creep etc. The thermal loading will increase the mechanical compressive stresses in the wing upper panels, and tend to reduce the mechanical tensile stresses in the wing lower panels. Thus, the thermo-mechanical buckling characteristics of the hot structural panels are a critical concern in the hypersonic aircraft wing structural panels. The effects of extreme environments with combined thermal and acoustic loading will require novel structural designs and the related understanding of the nonlinear response behavior. These extreme environments will contain large temperature gradients and random acoustic excitations with high sound pressure levels. The surfaces of these vehicles can, therefore, exhibit local buckling and post-buckling behavior under the dynamics excitation. One such post-buckling phenomenon is known as “snap-through” which can result due to combination of static and dynamic loading. The effect of snap-through is to increase the effective dynamic strain level. Future air vehicle structures will be designed to avoid local buckling and post-buckling phenomena including snap-through so these phenomena must be understood in greater detail. The current capabilities of the finite element method based design tools do not have satisfactory prediction capabilities for the post-buckling dynamic response behavior.
Under certain thermal-acoustic loading conditions, a dynamic instability will give rise to a snap-through response, which can significantly reduce fatigue life. The snap-through problem has previously been investigated using reduced-order analyses with both closed form and finite element (FE) [7-8, 35-40, 49, 50] solutions. In reduced-order FE analysis first the system is transformed to a reduced set of coupled nonlinear equations, which are solved via numerical integration. Since the eventual application is the analysis and design of practical structures, a significant improvement can be obtained using a basis consisting of both low-frequency transverse-dominated modes and high-frequency in-plane-dominated modes. Several response characteristics are demonstrated in the structures under investigation: (i) small amplitude vibration around one of two stable, buckled equilibrium positions, (ii) intermittent snap-through response between the two buckled equilibrium positions, and (iii) persistent snap-through response between the two buckled equilibrium positions. In each case, the reduced-order analysis may require different reduced order models.

**LITERATURE REVIEW**

Thermal buckling problems of single plates (continuous or laminated composites) were investigated by several authors including [2–11, 50-56], and thermo-mechanical buckling characteristics of hot structural sandwich panels were analyzed extensively by Ko and Jackson [20-25], using the minimum potential energy method. Dynamic buckling studies were limited to columns, but eventually, dynamic stability of thin-walled structures, i.e. plates and shells, received significant attention. Bolotin [2] illustrated the solution of a typical problem for isotropic structures. Dynamic stability of isotropic cylindrical shells was first investigated by Volmir [62] and Bolotin [2]. The first paper on dynamic stability of anisotropic shells was published by Markov [29]. The studies of dynamic stability of composite cylindrical shells were published by Ng [39].

One of the most interesting aspects of the post-buckled equilibrium behavior of flat plates and panels is that they may exhibit snap-through. This form of secondary bifurcation, or mode jumping, is associated with subtle interplay between modes and has received considerable attention [7-10, 17-19, 57-63]. After Stein’s [48] initial observation of such transient change in the post-buckled deformation states in a compression test of a multi-bay, flat aluminum plate, numerous analytical studies of the snap phenomenon and related issues have been performed on compressively loaded plates [49-54]. Linear eigenvalue analysis indicates that there exist many compound bifurcation points in the plots of critical loads versus the aspect ratio for various boundary conditions. Thus, it seems reasonable to assume that such critical loads are generated by a splitting process of the compound points by the variation of the aspect ratio from a particular value giving the multiple eigenvalues, and that the jump of the post-buckled modes caused by the secondary instability of the equilibrium can be explained by coupling effects between the competitive modes [58-60]. Many of the previous efforts exploring the post-buckling behavior and snap phenomena of axially compressed plates are carried out by using modal approximations, with the transverse deflection represented by a series of linear buckling modes [3–10, 14, 22–25]. However, as pointed out by Stoll [49-50], analyses using only two-term representations of the transverse displacements have failed to predict the snap phenomena of simply-supported plates [6, 9, 16]. With additional terms incorporated, or by using different approaches, the secondary instability is predicted [4, 5, 7, 10, 22]. The Analysis of a particular aspect ratio to a range of lengths has been achieved by Nakamura and Uetani [37] and Everall and Hunt [7-9, 17]. Nakamura and Uetani [37] systematically identified the consistent set of transverse deformation terms which is required to accurately predict the mode jumping for simply-supported plates over a range of lengths. Alternatively, Everall and Hunt comparatively studied the mode jumping in the buckling of struts and plates for a particular set of boundary conditions and presented their results by using the parametric space of Arnold’s tongues. Additional work includes [52-65].

Although in using approximate analytical techniques some insight can be obtained about the snap-through post-buckling phenomenon, finite element analysis is well suited to study this more complex behavior [3, 13, 15, 30-31, 36, 46, 50]. It is difficult to use the static path-following method to locate the disconnected stable equilibria before and after the jump. A standard path-following method is continued, after a convergence of a solution on the mode jumping path is obtained, to find the jumped path [15, 27]. Another is the well-known hybrid static–dynamic computational approach which is developed and used by Riks [42] to model one bay of Stein’s aluminum plate [48].

Finite element analysis can be used to predict response of structures under combined loading by integrating a set of full-order, nonlinear equations, but this is computationally prohibitive for design studies. Several methods exist
on the construction of reduced order models from finite element analysis [12-15, 30-34, 40-43]. The nonlinear, finite-element equations in physical degrees-of-freedom (DOF) are transformed to a lower order model in generalized DOF. The reduction is accomplished through transformation to generalized coordinates using a basis set which is often constructed from a truncated set of the normal mode shapes. Large amplitudes can cause transverse displacements to be nonlinearly coupled to membrane displacements (mid-surface stretching). Thus the basis set must also include vectors which span the space of membrane displacements. Membrane modes are much higher in frequency than the bending modes included in the model and are difficult to identify. An alternative approach implicitly incorporates the membrane effects into the bending stiffness terms through an estimation routine via the *implicit condensation* (IC) method. Previous studies in snap-through dynamics have looked at a variety of excitations including aerodynamic, acoustic, harmonic and stochastic [19, 32-33, 35, 39, 41, 43, 47, 62].

These reduced order models have a form which is known as Duffing-van der Pol oscillator which has been well studied in its local and global bifurcation behavior. In particular, it exhibits non-degenerate Hopf bifurcation. When the Duffing-van der Pol oscillator is perturbed by multiplicative white noise, it describes the model excited by inhomogeneities of the surrounding medium. Early approaches to the examination of the Hopf bifurcation behavior of the stochastic Duffing-van der Pol equation have been done. A joint work of Arnold, Sri Namachchivaya and Schenk-Hoppe [1] concerning the Duffing-van der Pol oscillator presents numerical results that are compared with the results obtained by applying the methods of stochastic normal form theory, asymptotic, and stochastic averaging on the description of stochastic bifurcations.

Further, the work of Simiu [47] is of interest as this provides the application of Melnikov’s method to the stochastically induced transitions in a spatially extended dynamical system. Melnikov based necessary conditions for the continuously buckled column for the occurrence of snap-through were obtained by Holmes and Marsdenn [9]. This was extended by Simiu [47] for the case of non-harmonic excitation, including random excitation. In this work, Melnikov necessary condition for chaos yields a simple criterion that guarantees the nonoccurrence of snap-through motion.

**POST-BUCKLING RESPONSE—SOME EXPERIMENTAL OBSERVATIONS**

The observation of the emergence of snap-through response is shown in Figure 1. These data were measured in the experiment described in [13, 15] but were not previously published. This experiment was designed so that a beam with no preload would have two symmetric bending mode natural frequencies in the range of 20-500 Hz. The beams were made from a high-carbon spring-steel with an effective length of 9.0 in, a nominal width of 0.5 in, and a nominal thickness of 0.031 in. The high-carbon steel material had a Young’s modulus of 29.7 Mpsi, a shear modulus of 11.6 Mpsi, and a mass density of $7.36 \times 10^{-4}$ lb-s$^2$/in$^4$. The test fixture was rigid compared to the beam. The coefficient of thermal expansion of the fixture material was matched as close to that of the beam as possible. Base excitation provided by a shaker was capable of applying distributed loads without otherwise affecting the beam’s mass, stiffness, or damping. A closed-loop shaker controller was used with an accelerometer on the shaker head to maintain the desired RMS input level and spectrum shape during testing. Time records of the displacement and surface strain at the beam mid-span, along with the excitation acceleration, were recorded. Three beam configurations were studied in the experiment: a beam without strain gages and no intentional preload, a second beam with strain gages attached and no preload, and finally, the same beam with strain gages and an induced tensile preload. The responses to five inertial load cases are presented for comparison: nominal RMS shaker table accelerations of 0.5, 1, 2, 4, and 8 g’s. Actual RMS input levels varied slightly from the nominal values. The input spectrum was flat between 20-500 Hz, and a sampling frequency of 4096 Hz was used throughout the testing. For the case with induced preload, the input spectrum was extended to 20-800 Hz while maintaining the RMS load value constant to capture the first two symmetric bending modes.

The data shown in Figure 1 were measured for the beam in a post-buckled state. The beam was removed from the clamping fixture and heated several degrees above room temperature. It was then quickly bolted into the fixture and the assembly was allowed to reach thermal equilibrium. The beam quickly buckled due to the thermal compressive stress. Figure 1 shows dynamic random response at the beam mid-point for three RMS input levels: 0.5g (nearly linear), 2g (mildly nonlinear) and 8g (strongly nonlinear). At the 0.5g input level, the response is relatively small and occurs about the static post-buckled position. As the input level is increased to 2g, the beam initially vibrates about the post-buckled position but then occasionally snaps to the other stable equilibrium position. Finally, at 8 g, the beam exhibits continuous oscillation between the two stable equilibrium positions.
Figure 1: Experimental snap-through displacement response of a clamped-clamped beam

REDUCED ORDER MODELING

The dynamic snap-through phenomenon is the jump between various modes of buckling. It should be noted that the effective stiffness of a structure undergoes a qualitative change as shown in Figure 2 via its load-deflection characteristics. Thus even though the magnitude of the deflection compared to the plate's dimension are not significant, this leads to a significant nonlinearity in the dynamic behavior of the plate. Under severe random acoustic excitation this snap-through phenomenon can happen very unpredictably and can cause fatigue. Typically, this phenomenon is represented by a modification of the Duffing oscillator, see [64] for details. The thermal loading modifies the large amplitude single mode plate motion to as given in Equation 1.
\[ \ddot{q} + \beta \dot{q} + k_0 (1-s)q + \alpha q^3 = f_0 + f(t) \]  

where the overhead dot denotes the time derivative, \( \beta \) is viscous damping, \( k_0 \) is the linear stiffness coefficient, \( s \) is the ratio of plate temperature to the buckling temperature, and \( \alpha \) is the nonlinear stiffness coefficient resulting from large displacements. The forcing terms \( f_0 \) and \( f(t) \) represent the static thermal loading due to the temperature gradient across the plate thickness and the dynamic loading respectively. The parameter \( s \) is a result of thermal expansion. The linear stiffness, \( k_0(1-s) \), consists of a negative thermal stiffness, \( -k_0s \), and a positive structural stiffness, \( k_0 \). The cubic term represents the geometric nonlinearity due to membrane stretching. The net potential energy of the overall system is represented by Equation 2.

\[ U(q) = f_0q + k_0 (1-s)q^2/2 + \alpha q^4/2 \]  

**NUMERICAL RESULTS – PERIODIC SOLUTIONS**

This potential energy is shown in Figure 3 and 4 as \( s \) and \( f_0 \) are varied respectively (where \( \alpha = k_0=1 \)). It should be noted that the effect of changing \( s \) results in a qualitative change in the system in terms of the potential well. For \( s < 1 \), the system exhibits a single well and as \( s \) exceeds unity two wells are formed. The space of displacement and \( s \) has a saddle. It should also be noted that the distance between the potential well increases as the parameter \( s \) increases.

This system is known to exhibit significant nonlinear behavior due to the interaction of the preload and nonlinear stiffness. The phase portrait of the post-buckled, un-damped system with pre-load is shown in Figure 5a and the associated time domain initial condition response is shown in Figure 5b. It should be noted that the snap-through motion can be observed leading to large strains and hence causing low-cycle fatigue response. For periodic excitation it can be shown that the effect of change in excitation amplitude is to result in significant nonlinear phenomenon including period doubling and chaotic response behavior. A numerical bifurcation diagram is shown in Figure 6. It should be noted that the first period doubling sequence leads to the snap-through motion.

A grid of points in the parameter space of preload, excitation amplitude and excitation frequency were analyzed using the continuation methods and the bifurcation boundary for snap-through motion was calculated. This boundary is shown in Figure 7 (a and b). The two parameters are as marked in the respective figures and the their axis denotes the variance in the Poincare points which can be used to highlight the first period doubling as a emergence of snap-through behavior. It should be noted that for certain values of excitation frequency snap-through can never be achieved (Figure 7a). Further, for increasing preload greater excitation amplitude is required to achieve snap-through (Figure 7b).

![Figure 3: Potential Energy as a function of displacement and the s, factor related to thermal buckling.](image-url)
Figure 4: Effect of a nonzero thermal loading is to create asymmetric wells in the potential energy of the system.

Figure 5a: Phase portrait of the undamped, post-buckled system with pre-load.

Figure 5b: Time response of the undamped, post-buckled system with pre-load exhibits snap-through.
Figure 6: Bifurcation diagram demonstrating the nonlinear phenomenon under periodic excitation – period doubling route to chaos is well known for this system.

Figure 7a: Numerical snap-through boundary in the parameter space of excitation frequency and amplitude.

Figure 7b: Numerical snap-through boundary in the parameter space of excitation amplitude and pre-load.
CONCLUSIONS

In this paper a detailed literature review is presented on the historical context and the state-of-the-art as applicable to the dynamic snap-through of structures. Further, some observations about the single frequency periodic excitation on the reduced-order models of the post-buckling dynamics is presented. In conclusion, the following are identifying features of the snap-through phenomenon, which is in essence a post-bifurcation dynamical property of the structures under random combined loading:

- It is a saddle-node bifurcation
- Beating is a non-stationary pre-cursor to jump
- Mode jumping (or snap-through) is associated with secondary instability
- First-period doubling sequence is a pre-cursor to escape from well.

The future direction of work in this area includes:
1. Development of linear matrix inequality formulation for snap-through prediction.
2. Development of snap-through boundary for randomly excited multi-mode reduced order models.
3. Computation of the closest distance to bifurcation.
4. Conduct parametric sensitivity analysis (using left and right eigenvectors) to identify design enhancement directions in parametric space.

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