The Frequency Based Substructuring (FBS) Method reformulated according to the Dual Domain Decomposition Method

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NOMENCLATURE

$N$ number of substructures
$u, U$ set of degrees of freedom in the time domain and in the frequency domain.
$M, C, K$ mass, damping and stiffness matrix of a linear(ized) system
$f, F$ external forces applied to the system in the time domain and in the frequency domain.
$\lambda$ Lagrange Multiplier expressing a connection force
$\omega$ frequency
$Z$ Dynamic stiffness matrix
$Y$ receptance matrix
$B$ Boolean operator expressing the compatibility constraints on an interface
$\ast^{(s)}$ pertaining to a substructure numbered $s$
$\ast^{A,B,...,Z}$ pertaining to a partitioned set of degrees of freedom
$\ast^T$ transpose of a matrix
$DOF$ degrees of freedom
$FRF$ Frequency Response Function
$A, B, C, I, J$ subset of DOF’s

ABSTRACT

In this article the Lagrange Multiplier Frequency Based Substructuring method (LM FBS) is introduced. This method is an evolution of the classical Frequency Based Substructuring (FBS) method where substructures characterized by their dynamic admittance can be assembled. In this paper we place the FBS method in the framework of dual Domain Decomposition methods in order to introduce in the problem the interface forces explicitly and in a general manner. In the LM FBS method Lagrange Multipliers are defined, which function as coupling forces between the substructure interfaces when they are coupled. With this formulation only boolean constraint matrices need to be defined. The resulting equations are easy to interpret and they are easy to implement in a computational environment. The proposed formulation of the FBS assembly opens new perspective for future improvement of experimental substructuring.

In this article, the LM FBS method is compared to other FBS methods. Here the similarity between the methods are illustrated and the advantages of the LM FBS method are outlined. We also indicate how the LM FBS Method can be reformulated to extract substructure data when the dynamics of the total structure is known.

Keywords: Dynamic Substructuring, Frequency Based Substructuring (FBS), Dual Domain Decomposition.

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1 INTRODUCTION

Dynamic Substructuring methods consist in dividing a system in subparts that can be analyzed separately then combing them together by an assembly procedure. Such methods were first introduced four decades ago in order to reduce the complexity of dynamical models and to reduce the size of computational models\textsuperscript{1,2,3}.

The dynamic substructuring methods have been used extensively in the past and many variants have been proposed over the years (see e.g.\textsuperscript{4,5,6,7,8,9,10,11}).

Although computer power has tremendously improved over the year allowing solving large problems and handling complex models, substructuring techniques are still very popular in engineering since they allow spreading the development work amongst different subgroups. Also, the models become more and more complex (in terms of number of degrees of freedom and in terms of physics modeled) reduction techniques are still necessary for instance when optimizing designs. The concept of substructuring is strongly related to domain decomposition methods which have become the corner stone of efficient parallel computing\textsuperscript{12}.

Different methods of Dynamic Substructuring exist. Two different classes of substructuring methods can be distinguished\textsuperscript{13}:

- Time-domain based methods,
- Frequency-domain based methods.

For the time domain based methods, each subsystem is described by a generalized mass, damping and stiffness matrix. In particular when the generalized substructure matrices are build using local modal properties one calls them Component Mode Synthesis (CMS). The modal synthesis technique determines the dynamic behavior of a coupled system on the basis of a normal mode description of the uncoupled systems. The most well known CMS technique is the Craig-Bampton method\textsuperscript{3}.

For the frequency domain based methods on the other hand, each subsystem is described in terms of Frequency Response Functions (FRF’s) of the uncoupled systems. This class is named Frequency Based Substructuring (FBS).

Modal synthesis methods are easy to implement whenever mass and stiffness matrices of the substructures are known theoretically, e.g. by a finite element model. However they are difficult to apply when dealing with experimental data. If modal synthesis methods are applied on experimental data, an identification technique has to be used in order to be able to determine the mass, damping and stiffness matrices of the subsystems. When experimental data are considered, FBS has some advantages:

- It uses the measured frequency response functions directly, which implies that errors introduced by modal analysis and the errors caused by high mode truncation are eliminated.
- Because the measured data represents the actual physical behavior of the structure, dependency of the structural dynamics on frequency (such as for visco-elastic materials) are included in the FRF measurements whereas they cannot be described by classical model synthesis approaches.

Experimental substructuring based on Frequency Based Substructuring approaches have become an important research issue in the last years\textsuperscript{14,15,16,17,18}. The advantages of experimental substructuring are numerous:

- It gives the possibility to combine modeled parts from either theoretical or numerical analysis, and measured components derived from experimental tests. Combining experimental and theoretical models is also referred to as hybrid analysis.
- The effect of changing the properties of a subsystem on the assembled system can be analyzed efficiently. Also by analyzing the subsystems, local dynamic behavior can be recognized more easily than when the entire system is analyzed.
• It allows sharing and combining of substructures from different project groups.

• When a substructure is changed, dynamic substructuring allows rapid evaluation of the dynamics of the complete system. Only the changed subpart needs to be measured and thereby allows efficient local optimization, fast design cycles and subsequently an overall optimization.

• Dynamic Substructuring can be convenient if a measurement cannot be done because the structure is too large or complex to be measured as a whole or if not enough excitation energy can be put in the structure for adequate excitation.

• It allows easier spotting of local problems that might not be visible by testing the entire structure.

Dynamic Substructuring also has some disadvantages. The main disadvantages are:

• Applicability of Dynamic Substructuring is usually limited to linear and stationary systems with constant parameters.

• For experimental substructuring, most measurements are limited to translational degrees of freedom because rotational degrees of freedom are difficult to measure. Assembling rotational dofs is thus a major challenge.

• Dynamic substructuring code can take substantial time to program.

• For experimental substructuring, measurements containing noise are used. The matrix inversion(s) that are needed in the algorithm(s) will propagate measurement noise, resulting in an inaccurate solution for the complete system.

Tackling these issues is essential if FBS techniques are to become the methods of choice for efficient experimental analysis of structures in the future. One important drawback in improving the FBS method is the unnecessary complexity involved in all publications relative to the subject. Indeed, using proper dual formulations the mechanical interpretation of FBS as well as the mathematical formulation can be greatly simplified, opening new opportunities for future breakthroughs in experimental substructuring.

In this article the Lagrange Multiplier Frequency Based Substructuring Method (LM FBS) will be introduced. This method is a reformulation and generalization of the classical FBS method. Section 2 summarizes the classic FBS method and section 3 sets out the LM FBS method. The comparison of the LM FBS method to other dynamic substructuring methods and conclusions are given in sections 4 and 5.

2 FREQUENCY BASED SUBSTRUCTURING

The objective of Frequency Based Substructuring (FBS) is to predict the dynamic behavior of a system made of subparts on the basis of free-interface frequency response functions (FRF) of the uncoupled substructures. The FRF are in the assembly procedure represent the structural dynamic stiffness between discrete points of the subsystems in the frequency domain. They can either be determined theoretically or experimentally.

2.1 Basic theory of FBS

Before discussing the methodology of FBS, the basic theory of decomposed equations of motion of a subsystem \(s\) in the frequency domain is recalled. The FRF of the subsystem can be determined theoretically by considering the dynamic equations of motion in time domain of the subsystem:

\[
M^{(s)} \dddot{u}^{(s)}(t) + C^{(s)} \dot{u}^{(s)}(t) + K^{(s)} u^{(s)}(t) = f^{(s)}(t)
\]  

(1)

In equation (1), \(M^{(s)}\), \(C^{(s)}\) and \(K^{(s)}\) represent the mass, damping and stiffness matrices of subsystem \(s\) respectively. \(f^{(s)}\) represents the force excitation vector of the subsystem.

Using a Fourier transformation, the dynamic behavior of the subsystem can be expressed in the frequency domain. The format of the equations of motion in the frequency domain depends on the physical quantity of the response parameter.
(i.e. displacement, velocity or acceleration) that is used in the analysis. Classically the equation of motion is written in
displacement, which leads to:
\[
\begin{bmatrix}
-\omega^2 M^{(s)} + j \omega C^{(s)} + K^{(s)}
\end{bmatrix} U^{(s)}(\omega) = F^{(s)}(\omega),
\]
where \( j = \sqrt{-1} \). Equation (2) can be rewritten as:
\[
Z^{(s)}(\omega) U^{(s)}(\omega) = F^{(s)}(\omega)
\]
where
\[
Z^{(s)}(\omega) = \begin{bmatrix}
-\omega^2 M^{(s)} + j \omega C^{(s)} + K^{(s)}
\end{bmatrix}.
\]
represents the Dynamic Stiffness matrix\(^{21}\) of subsystem \( s \). Equation (3) can be rewritten as
\[
Y^{(s)}(\omega) F^{(s)}(\omega) = U^{(s)}(\omega)
\]
where \( Y^{(s)}(\omega) \), the inverse of the dynamic stiffness matrix \( Z^{(s)}(\omega) \), is the so-called receptance matrix of subsystem \( s \). In
subsequent parts of this article the dynamic stiffness and receptance matrices of a substructure \( s \) will be written as \( Z^{(s)} \) and \( Y^{(s)} \) respectively and it is implied that \( Z^{(s)} \) and \( Y^{(s)} \) are frequency dependent.

The receptance matrices are used in the frequency based substructuring method by Jetmundsen et al.\(^{22}\) to couple the
substructures. The theory utilizes graph theory, which provides a useful tool to couple arbitrary subsystems in a systematic
way. Also the use of matrix partitioning simplifies the coupling process. Using the original notation of Jetmundsen\(^{20,22}\)
the receptance matrices of the two subsystems shown in figure 1 are written as follows:
\[
\begin{pmatrix}
U^{(a)}_A \\
U^{(a)}_I \\
U^{(b)}_B
\end{pmatrix}
= \begin{bmatrix}
Y^{(a)}_{AA} & Y^{(a)}_{AI} & Y^{(a)}_{AB} \\
Y^{(a)}_{IA} & Y^{(a)}_{II} & Y^{(a)}_{IB} \\
Y^{(b)}_{BA} & Y^{(b)}_{BI} & Y^{(b)}_{BB}
\end{bmatrix}
\begin{pmatrix}
F^{(a)}_A \\
F^{(a)}_I \\
F^{(b)}_B
\end{pmatrix}
= [Y]^{(a)} \begin{pmatrix}
F^{(a)}_A \\
F^{(a)}_I \\
F^{(b)}_B
\end{pmatrix}
\]
\[
\begin{pmatrix}
U^{(b)}_A \\
U^{(b)}_I \\
U^{(b)}_B
\end{pmatrix}
= \begin{bmatrix}
Y^{(b)}_{AA} & Y^{(b)}_{AI} & Y^{(b)}_{AB} \\
Y^{(b)}_{IA} & Y^{(b)}_{II} & Y^{(b)}_{IB} \\
Y^{(b)}_{BA} & Y^{(b)}_{BI} & Y^{(b)}_{BB}
\end{bmatrix}
\begin{pmatrix}
F^{(b)}_A \\
F^{(b)}_I \\
F^{(b)}_B
\end{pmatrix}
= [Y]^{(b)} \begin{pmatrix}
F^{(b)}_A \\
F^{(b)}_I \\
F^{(b)}_B
\end{pmatrix}
\]

![Figure 1: Coupling of subsystems 'a' and 'b'.](image)

Figure 1 shows that the system consists of the subsystems 'a' and 'b'. Both subsystems have internal degrees of freedom,
represented by sets 'A' and 'B' respectively. The interface degrees of freedom are represented by 'I' and are shared by
both subsystems. The assembled system 'ab' is constructed when the subsystems 'a' and 'b' are coupled. In the coupling
procedure, the set of interface degrees of freedom become internal degrees of freedom of the system 'ab' so that the
assembled system is governed by
\[
\begin{pmatrix}
U^{(ab)}_A \\
U^{(ab)}_I \\
U^{(ab)}_B
\end{pmatrix}
= \begin{bmatrix}
Y^{(ab)}_{AA} & Y^{(ab)}_{AI} & Y^{(ab)}_{AB} \\
Y^{(ab)}_{IA} & Y^{(ab)}_{II} & Y^{(ab)}_{IB} \\
Y^{(ab)}_{BA} & Y^{(ab)}_{BI} & Y^{(ab)}_{BB}
\end{bmatrix}
\begin{pmatrix}
F^{(ab)}_A \\
F^{(ab)}_I \\
F^{(ab)}_B
\end{pmatrix}
= [Y]^{(ab)} \begin{pmatrix}
F^{(ab)}_A \\
F^{(ab)}_I \\
F^{(ab)}_B
\end{pmatrix}
\]
\[
\begin{pmatrix}
U^{(ab)}_A \\
U^{(ab)}_I \\
U^{(ab)}_B
\end{pmatrix}
= \begin{bmatrix}
Y^{(ab)}_{AA} & Y^{(ab)}_{AI} & Y^{(ab)}_{AB} \\
Y^{(ab)}_{IA} & Y^{(ab)}_{II} & Y^{(ab)}_{IB} \\
Y^{(ab)}_{BA} & Y^{(ab)}_{BI} & Y^{(ab)}_{BB}
\end{bmatrix}
\begin{pmatrix}
F^{(ab)}_A \\
F^{(ab)}_I \\
F^{(ab)}_B
\end{pmatrix}
= [Y]^{(ab)} \begin{pmatrix}
F^{(ab)}_A \\
F^{(ab)}_I \\
F^{(ab)}_B
\end{pmatrix}
\]
In equation (8) \([\mathbf{Y}]_{ab}\) represents the receptance matrix of the assembled system. The forces in vector \(\mathbf{F}^{(ab)}\) represent external forces which are applied to the former interface degrees of freedom. The receptance matrix \(\mathbf{Y}^{(ab)}\) in equation (8) cannot be determined directly from the receptance matrices of the subsystems. Imposing that the interface degrees of freedom must be equal (compatibility condition) and that the internal interface forces are equal on opposite on reciprocating DOFs, it can be shown that (see e.g. 13,15,17)

\[
[\mathbf{Y}]^{ab} = \begin{bmatrix}
\mathbf{Y}^{(a)}_{AA} & \mathbf{Y}^{(a)}_{AI} & 0 \\
\mathbf{Y}^{(a)}_{IA} & \mathbf{Y}^{(a)}_{II} & 0 \\
0 & 0 & \mathbf{Y}^{(b)}_{BB}
\end{bmatrix} - \begin{bmatrix}
\mathbf{Y}^{(a)}_{AI} \\
\mathbf{Y}^{(a)}_{II} \\
-\mathbf{Y}^{(b)}_{BI}
\end{bmatrix} (\mathbf{Y}^{(a)}_{II} + \mathbf{Y}^{(a)}_{II})^{-1} \begin{bmatrix}
\mathbf{Y}^{(a)}_{AI} \\
\mathbf{Y}^{(a)}_{II} \\
-\mathbf{Y}^{(b)}_{BI}
\end{bmatrix}^T
\]  

Note that the interface DOF’s of subsystem ‘a’ were retained, whereas also the interface DOF’s of subsystem ‘b’ could have been chosen. This choice will lead to a different expression for the total system ‘ab’. Equation (10) represents the receptance expression of the assembled system, which was first introduced by Jetmundsen et al. 20,22,23.

With the previously defined equations in mind, the theory can be generalized for an arbitrary number of subsystems and couplings 14,20. Using matrix partitioning and the graph theory framework the generalized receptance coupling equation for an arbitrary number of subsystems can be written as:

\[
[\mathbf{Y}'] = [\mathbf{Y}_{\alpha\alpha}] - ([\mathbf{M}] \circ [\mathbf{Y}_{\alpha\gamma}])^{-1} \sum_{s=1}^{N} [\mathbf{B}_s] \circ [\mathbf{Y}_{s\gamma}]^{-1} [\mathbf{M}] \circ [\mathbf{Y}_{\alpha\gamma}]^T
\]

where:

- \(\alpha\) = Sets of internal and interface degrees of freedom of the subsystems.
- \(\gamma\) = Sets of interface degrees of freedom of subsystems.
- \([\mathbf{Y}']\) = The total synthesized receptance matrix.
- \([\mathbf{Y}_{\alpha\alpha}]\) = The receptance matrix of the uncoupled subsystems relative to \(\alpha\).
- \([\mathbf{M}]\) = The boolean mapping matrix.
- \([\mathbf{Y}_{\alpha\gamma}]\) = The receptance matrix with respect to internal and interface degrees of freedom.
- \([\mathbf{Y}_{s\gamma}]\) = The receptance matrix with respect to the interface degrees of freedom.
- \(\mathbf{B}_s\) = The boolean interface matrix for the \(s\)th subsystem.
- \(N\) = The number of subsystems.
- \(\circ\) = The Hadamard or element by element matrix product.

The Boolean mapping matrix \([\mathbf{M}]\) and the Boolean interface matrices \(\mathbf{B}_s\) define how the subsystems are interconnected. The Boolean interface matrix can be calculated out of the Boolean mapping matrix. In total \(N + 1\) boolean matrices are needed for the assembly of \(N\) subsystems.

Note that in the quite intricate formulation (11) the analyst has to choose which interface DOF’s are retained in the calculation, because the interface DOF of only one subsystem can be retained. This choice has to be made preliminary to the FBS calculation.

Although the Jedmundsen formulation is able to handle the coupling between arbitrary systems, it is fairly complex to use in practice. The LM FBS method will allow to simplify the assembly of FRF’s and allow a direct mechanical interpretation of the procedure.

3 LAGRANGE MULTIPLIER FREQUENCY BASED SUBSTRUCTURING

In this section we will show that the concept of dual assembly of structure as used in parallel computation 12) and model reduction 11 can allow to assemble FRFs in an elegant way.
3.1 Boolean mapping matrix

The Boolean Mapping Matrix of a system of subsystems contains information about the manner in which the subsystems are coupled. Consider the three subsystems shown in figure 2. The system consists of three subsystems 'a', 'b' and 'c'. The internal degrees of freedom are represented by 'A', 'B' and 'C'. The interface degrees of freedom are represented by 'I' and 'J'.

The compatibility condition imposing that the degrees of freedom on the interface have to be equal along interface I and J can be written as

\[ BU = 0 \]  

where

\[ B = \begin{bmatrix} A & I^{(a)} & B & I^{(b)} & J^{(b)} & C & J^{(c)} \\ 0 & -I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & I \end{bmatrix} \]  

\[ U = \begin{bmatrix} U_A^{(a)} \\ U_B^{(a)} \\ U_B^{(b)} \\ U_C^{(b)} \\ U_J^{(b)} \\ U_J^{(b)} \\ U_J^{(c)} \end{bmatrix} \]  

is a signed Boolean constraint matrix and

The constraint equation (12) between the subsystem DOFs expresses the compatibility condition on the interface. For instance (see figure 2) on the interface 'I' related to the first set in (12) we have

\[ U_J^{(b)} - U_I^{(a)} = 0. \]  

We will now indicate how these constraints can be applied to assemble the subsystems in one structure.

3.2 Dual assembly

With LM FBS, the matrix of the coupled system is assembled in a dual manner (also called an augmentation formulation in the field of multibody dynamics). The compatibility condition on the interface will be juxtaposed to the subsystem
dynamic equilibrium equation (3). Additional unknown interface forces are then introduced to enforce the compatibility constraints so that the local equilibrium equations can be written as

\[ ZU + B^T \lambda = F \]  

(16)

where, using a partition per substructure (for the example of figure 2),

\[ Z = \begin{bmatrix} Z^{(a)} & 0 & 0 \\ 0 & Z^{(b)} & 0 \\ 0 & 0 & Z^{(c)} \end{bmatrix}, \quad U = \begin{bmatrix} U^{(a)} \\ U^{(b)} \\ U^{(c)} \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_I \\ \lambda_J \end{bmatrix} \]  

(17)

In equation (16) the operator \( B^T \), the transposed of (13), defines on which DOFs the connecting forces \( \lambda \) will be applied. \( \lambda \) are in fact the Lagrange multipliers associated to the constraints (12).

Combining the dynamic equilibrium equations (16) with the compatibility equations (12) the behavior of the complete system can be described as

\[ \begin{bmatrix} Z \quad B^T \\ B \quad 0 \end{bmatrix} \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}. \]  

(18)

As an example the system of equations (18) is written explicitly for the system shown in figure 2:

\[ \begin{bmatrix} Z^{(abc)} \\ B \end{bmatrix} B^T = \begin{bmatrix} Z^{(a)} & Z^{(a)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Z^{(a)} & Z^{(a)} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z^{(b)} & Z^{(b)} & Z^{(b)} & 0 & 0 & 0 \\ 0 & 0 & Z^{(b)} & Z^{(b)} & Z^{(b)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z^{(c)} & Z^{(c)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z^{(c)} & Z^{(c)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U \quad \lambda \end{bmatrix} = \begin{bmatrix} F^{(abc)} \\ 0 \end{bmatrix} \]  

(19)

Note that the format (i.e. the following order of the degrees of freedom) regarding the internal and interface degrees of freedom of the impedance matrices and the Boolean mapping matrix is arbitrary.

Solving the local equilibrium equations in (18) for \( \lambda \) one finds

\[ U = Y(F - B^T \lambda) \]  

(20)

and substituting in the second set of equations of (18) representing the compatibility equations, one obtains

\[ BY(F - B^T \lambda) = 0 \]

\[ BYB^T \lambda = BYF \]  

(21)
where

\[
Y = \begin{bmatrix}
Y^{(a)} & 0 & 0 \\
0 & Y^{(b)} & 0 \\
0 & 0 & Y^{(c)} \\
\end{bmatrix} = \begin{bmatrix}
Z^{(a)^{-1}} & 0 & 0 \\
0 & Z^{(b)^{-1}} & 0 \\
0 & 0 & Z^{(c)^{-1}} \\
\end{bmatrix} = Z^{-1}
\] (22)

Equation (21) can be solved for \( \lambda \) and substituting into equation (20) one obtains

\[
U = Y(I - B^T(BYB^T)^{-1}BY)F
\] (23)

which is equivalent to the classical FBS method according to Jetmundsen, as will be shown in section 4.

3.3 Fictitious Domain Substructuring

For many complex problems, the FRF’s of the individual substructures may not be easily obtained. Especially when substructures, like rubber mountings for example, are coupled with a preload, the measurement of the substructure alone is difficult to perform. Furthermore, preloads, temperature influences and other nonlinear effects might not be known in advance\(^\text{17}\). It is therefore interesting to extract the subsystem FRF’s out of the measurement of the total system.

Let’s assume for example that the receptance matrix of substructure ‘c’ is wanted (see figure 3), meaning that we want to represent the dynamics of subsystem ‘c’ is uncoupled. This can be obtained as follows when the FRF’s of the entire structure ‘abc’ and the subsystem ‘ab’ (see figure 4) are known.

This problem can be reformulated as finding the response of the part ‘c’ of the full system ‘abc’ when additional forces \( B^T_I \lambda \) are applied on interface \( I \) such that total forces on \( I \) are null. That means that the forces \( B^T_I \lambda \) have to be equal to the forces produced on the interface by the subsystem ‘ab’. Mathematically this means that the response of uncoupled subsystem ‘c’ is solution of

\[\text{Figure 3: Substructure of interest.}\]

\[\text{Figure 4: Measured systems.}\]
Figure 5: Scheme for fictitious domain substructuring.

\[
\begin{align*}
Z^{(abc)}U^{(abc)} + B^{(abc)}_T \lambda &= F^{(c)} \\
Z^{(ab)}U^{(ab)} - B^{(ab)}_T \lambda &= 0 \\
B^{(abc)}_T U^{(abc)} + B^{(ab)} T U^{(ab)} &= 0,
\end{align*}
\]

where the boolean matrix \( B \) from equation (18) is partitioned in \( B^{(abc)} \) and \( B^{(ab)} \). In matrix notation one can write

\[
\begin{bmatrix}
Z^{(abc)} & 0 & B^{(abc)}_T \\
0 & Z^{(ab)} & -B^{(ab)}_T \\
B^{(abc)}_T & B^{(ab)} & 0
\end{bmatrix}
\begin{bmatrix}
U^{(abc)} \\
U^{(ab)} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
F^{(c)} \\
0 \\
0
\end{bmatrix}.
\]

This system is non-symmetric. It becomes symmetric if the substructure 'ab' is multiplied by \(-1\):

\[
\begin{bmatrix}
Z^{(abc)} & 0 & B^{(abc)}_T \\
0 & -Z^{(ab)} & -B^{(ab)}_T \\
B^{(abc)}_T & B^{(ab)} & 0
\end{bmatrix}
\begin{bmatrix}
U^{(abc)} \\
U^{(ab)} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
F^{(c)} \\
0 \\
0
\end{bmatrix}.
\]

This last relation clearly shows that the extraction of a subsystem out of a total system is equivalent to a dual assembly of a negative dynamic stiffness for the substructure that we want to subtract (in this example substructure ‘ab’). In general one can write

\[
\begin{bmatrix}
Z^{(tot)} & 0 & B^{(tot)}_T \\
0 & -Z^{(fict)} & B^{(fict)}_T \\
B^{(tot)}_T & B^{(fict)} & 0
\end{bmatrix}
\begin{bmatrix}
U^{(tot)} \\
U^{(fict)} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
F^{(sub)} \\
0 \\
0
\end{bmatrix},
\]

where the measurement of the total arbitrary system is stored in the \( Z^{(tot)} \), the subsystems which are subtracted from the total system are stored in the \( Z^{(fict)} \) and the load, which is applied to the remaining substructure, are addressed in the force \( F^{(sub)} \). Here the term ‘fict’ stands for fictitious domain, which is used in the Dual Domain Decomposition method (see e.g. 24).

4 COMPARISON OF LM FBS WITH THE CLASSIC FBS METHOD

The Jetmundsen method is an elegant method; it allows coupling substructures in a systematic way and is therefore suitable to be used in a computational environment. As mentioned before, the Jetmundsen method has been a source
of inspiration for the LM FBS method. LM FBS uses the same advantages and integrates new ones, obtaining the same results as with the method of Jetmundsen et al. In this section the LM FBS Method is compared to the Jetmundsen method. It is shown that the methods are equivalent (which is natural since they describe the same dynamics of the assembled system).

The coupled receptance expression assembled with the method of Jetmundsen for the system shown in figure 2 is given by

\[
\begin{bmatrix}
    Y_{(abc)}^{(ab)} \\
    Y_{(abc)}^{(ac)} \\
    Y_{(abc)}^{(bc)}
\end{bmatrix}_{Jetmundsen} = 
\begin{bmatrix}
    Y_{AA}^{(a)} & Y_{AI}^{(a)} & 0 & 0 & 0 & 0 \\
    0 & Y_{BB}^{(b)} & Y_{BJ}^{(b)} & 0 & 0 & 0 \\
    0 & 0 & Y_{BB}^{(b)} & Y_{BJ}^{(b)} & 0 & 0 \\
    0 & 0 & 0 & Y_{BB}^{(b)} & Y_{BJ}^{(b)} & 0 \\
    0 & 0 & 0 & 0 & Y_{CC}^{(c)} & 0 \\
    0 & 0 & 0 & 0 & 0 & Y_{CC}^{(c)}
\end{bmatrix} - \ldots
\]

where the interface degrees of freedom are retained for subsystem 'b'.

For the LM FBS method let us write (23) where we introduce the partitioning used for (19):

\[
\begin{bmatrix}
    Y_{(abc)}^{(ab)} \\
    Y_{(abc)}^{(ac)} \\
    Y_{(abc)}^{(bc)}
\end{bmatrix}_{LMFBS} = 
\begin{bmatrix}
    Y_{AA}^{(a)} & Y_{AI}^{(a)} & 0 & 0 & 0 & 0 \\
    0 & Y_{AI}^{(a)} & Y_{II}^{(a)} & 0 & 0 & 0 \\
    0 & 0 & Y_{II}^{(a)} & Y_{IJ}^{(a)} & 0 & 0 \\
    0 & 0 & 0 & Y_{II}^{(a)} & Y_{JJ}^{(a)} & 0 \\
    0 & 0 & 0 & 0 & Y_{CC}^{(c)} & 0 \\
    0 & 0 & 0 & 0 & 0 & Y_{CC}^{(c)}
\end{bmatrix} - \ldots
\]

Looking at equations (30) and (31), it is obvious that the equations are not exactly the same. To analyze the equivalence of the two forms, let us observe that equation (31) contains the receptances with respect to the interface degrees of freedom 'I(a)', 'I(b)', 'J(b)' and 'J(c)'. Due to the coupling process we obviously have that 'I(a) = I(b)' and 'J(b) = J(c)'. This implies that only two of the four coupling degrees of freedom are independent in (31). In the Jedmunsen formulation only the interface DOF's of subsystem 'b' is present. So to compare the formulation we can drop in (31) the redundant equations related to 'I(a)' and 'J(c)', namely the second and seventh line in (31). Finally one then has to show that the first and last set of equations in (30) are respectively equivalent to the first and sixth set of equations in (31).

In order to simplify the discussion and to show the equivalence for a simple case, let us assume that the structure is made of subsystems 'a' and 'b' only. We then only have the interface set I so that the dynamic equation for the assembled 'ab'
which is obviously satisfied.

Next let us show that the first line of the Jetmundsen equation (32) is equivalent to the first line of the LM FBS equation (33) respectively related to $U_i^{(a)}$ and $U_i^{(b)}$ are identical. This can be proved by taking the difference of those sets and find

$$
\begin{bmatrix}
Y_{IA}^{(a)} & 0 \\
0 & Y_{IB}^{(a)}
\end{bmatrix}
+ Y_{II}^{(a)} \begin{bmatrix}
Y_{II}^{(a)} & 0 \\
0 & Y_{II}^{(a)}
\end{bmatrix}^{-1}
= 0
$$

Next let us show that the first line of the Jetmundsen equation (32) is equivalent to the first line of the LM FBS equation (33). Comparing the first sets in (32) and (33) using the partitioning (34) and (35) we see that the equality holds if

$$
- Y_{AI}^{(a)} \begin{bmatrix}
Y_{II}^{(a)} & 0 \\
0 & Y_{II}^{(a)}
\end{bmatrix}^{-1}
= 0
$$

Simplifying this relation we find

$$
0 = Y_{AI}^{(a)} \begin{bmatrix}
Y_{II}^{(a)} & 0 \\
0 & Y_{II}^{(a)}
\end{bmatrix}^{-1}
\begin{bmatrix}
F^{(a)}_T + F^{(b)}_T \\
-F^{(a)}_T + F^{(b)}_T
\end{bmatrix}
- Y_{AI}^{(a)} \begin{bmatrix}
Y_{II}^{(a)} & 0 \\
0 & Y_{II}^{(a)}
\end{bmatrix}^{-1}
\begin{bmatrix}
F^{(a)}_T + F^{(b)}_T \\
-F^{(a)}_T + F^{(b)}_T
\end{bmatrix}
$$

which is obviously satisfied.
Advantages of the LM FBS method over the classic method are:

In this article a reformulation of the classic FBS Method was introduced, which carries the name Lagrange Multiplier.

5 CONCLUSION

Equation (39) represents terms from the Jetmundsen coupled receptance with different interface degrees of freedom. When equations (30) and (39) are compared, it shows that different off-diagonal submatrices of the subsystem FRF’s are used.

Let us finally note that when the LM FBS method is applied, iterative solvers can be used to solve (21) (see e.g.\(^{12}\)). Such solvers are very efficient and naturally parallel for the dual interface problem. Moreover, tuning the tolerance in the solution process and using proper preconditioners iterative solvers could allow to filter the solution in order to reject spurious components of the assembled solution arising from measurement errors in the subsystems. This is a issue for further investigation.

5 CONCLUSION

In this article a reformulation of the classic FBS Method was introduced, which carries the name Lagrange Multiplier Frequency Based Substructuring (LM FBS). It was shown that the LM FBS method is equivalent to the classical theory. Advantages of the LM FBS method over the classic method are:

- The formulation of LM FBS is easier to write and interpret compared to the classic method of Jetmundsen and others.\(^{17,25,26}\).
- Only one Boolean matrix is needed to couple the \(N\) substructures. With Jetmundsen \(N + 1\) Boolean matrices are needed.
• The engineer can choose which FRF data is used for the interface DOF. Here the FRF’s of substructure 1, 2 or combinations between both can be used, to optimize the end result.

• The LM FBS method can also be used to extract substructures out of a total system. The extraction succeeds through a normal assembly with a negative dynamic stiffness matrix for the substructure which needs to be subtracted.

REFERENCES


