ROBUST MODAL SYNTHESIS IN DYNAMIC STOCHASTIC

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NOMENCLATURE AND MAIN SYMBOLS

DOF Degrees of Freedom
MP Modal Perturbation Method
MEF Finite Elements Model
SFEM Stochastic Finite Elements Method
$H(x,\theta)$ Random Field
$M_\theta , K_\theta$ Mean Mass and Stiffness Matrices
$M(\theta), K(\theta)$ Stochastic Mass and Stiffness Matrices
$Z_\theta$ Mean Dynamic Stiffness Matrix and Perturbation Matrix
$U(\theta)$ Random response vector
$T$ Condensation basis of model
$f_\Delta$ Localized Stochastic forces Vector
$f_{nl}(\hat{U}, U)$ Non-linear Stochastic forces vector

ABSTRACT

The study presented in this article concerns the robust modal synthesis of stochastic models in linear and non-linear elastodynamics. One can propose a strategy allowing the coupling of the stochastic finite elements method (SFEM) and a robust dynamic condensation method vis-à-vis of the stochastic parametric modifications in view to construct a robust reduced uncertain model.

This strategy is based on the exploitation of the (SFEM) that consists to combine the classic Finite Elements analysis and the statistical analysis. It is also based on the utilization of a dynamic condensation basis of Karhunen-Loève type enriched by the uncertain residuals static responses.

This strategy is extended to the uncertain structures presenting localized non-linearities at the interfaces between the sub-structures.

The exploitation of a perturbation methods based on a modal approach permits to obtain rapid modal synthesis of the uncertain response. The coupling of this approach with the robust condensation method allows the treatment of the complex structures of which the Finite Elements model is of large size.

Otherwise the difficulties linked to the calculation time, to the number of uncertain parameters, to the parametric modification levels and the degree of non-linearity will be highlighted to illustrate the advantages of the proposed methodology in comparison with the classic approaches.

The interest of the proposed method and its performances will be illustrated by a numerical simulation example.

1. INTRODUCTION

The predicted calculations of the mechanical systems or structures in the design phase or before the design phase present often differences between results of the mathematical models and tests. These differences come mainly from the complex phenomenon badly modeled (law of material behavior, friction, slipping and the junctions,....), the uncertainties on the geometric and mechanical characteristics (geometric parameters,
characteristics of the material, boundary conditions,...) and the non-linearities which are not taken into account at the interfaces between sub-structures.

The modal analysis is widespread at the analytical and the experimental levels in the study of the dynamic mechanical structure. However it remains often limited only to the linear cases.

Therefore, the extension of modal synthesis methods to the non-linear structures is very useful in the improvement of the survey of the dynamic behavior of the non-linear structures through the consideration of the various sources of non-linearities.

In addition the consideration of the influence of the modification parameters uncertainties permits to improve of advantage the prediction of the dynamic behavior of the linear and non-linear structures.

On the one hand, one can propose to exploit the SFEM that consists in regrouping the classic analysis by Finite Elements and the statistical analysis.

The stochastic methods permit to analyze the variability of the dynamic response of systems from the uncertain parameters properties of the structure.

Moreover, the stochastic analysis methods of the structures are generally classified in three categories; the Monte Carlo simulation methods that are often considered as of reference methods, the perturbation methods that are based on the expansion of Taylor series of the responses around the means of the uncertain variables [1] and the spectral methods that use basis functions of the Hilbert space associated to the uncertain problems; these functions are often the orthogonal polynomials, and the polynomial chaos in particular [2].

Recently, a modal perturbative approach [3] using some exact methods [4, 5], allows an efficacious calculation of the random eigenmodes and a fast synthesis of the uncertain frequency response in order to avoid the bad conditioning of matrices around the resonances.

One can obtain the means values of the extreme statistics and the standard deviation that give a good evaluation of the solution variability envelope.

With the aim of obtaining considerable reduction of the calculation costs and a good predictivity of non-linear model, one can propose a strategy permitting to couple the Stochastic Finite Elements Method (SFEM) and a robust dynamic condensation method vis-à-vis of the stochastic parametric modifications [6-8] in view to construct a robust reduced uncertain model. This Condensation method is based on the Karhunen-Loeve approach. This method permits to assign to zones or sub-structures specific uncertainties of weak, means or large level.

Besides, the extension of this robust condensation method to the case of the complex structures of which the Finite Elements Model is of large size and presenting a localized non-linearities allows the obtaining of a robust and fast modal synthesis for this type of structure.

2. NON-LINEAIR STOCHASTIC MODEL

The Karhunen-Loeve decomposition of the random field \( H(\cdot) \) on the basis of the eigenfunctions \( f_r(x) \) is given by:

\[
H(x,\theta) = \mu(x) + \sum_{r=1}^{q} \sqrt{\lambda_r} f_r(x) \xi_r(\theta)
\]

of truncated shape : \( \hat{H}(x,\theta) = \mu(x) + \sum_{r=1}^{q} \sqrt{\lambda_r} f_r(x) \xi_r(\theta) \)  \( (1) \)

Where \( \{\xi_r(\theta), r = 1,\ldots,q\} \) are the independent Gaussian standard normal variables and \( \{\lambda_r, f_r\} \) represent respectively the eigenvalues and eigenvectors of the known covariance function \( C(x_1, x_2) \) associated to \( H(\cdot) \), whose the spectral decomposition is written as:

\[
C(x_1, x_2) = \sum_{r=1}^{q} \lambda_r f_r(x_1) f_r(x_2)
\]

\( (2) \)

The Stochastic mass Matrix is the assembly of the elementary matrices:

\[
M(\theta) = \mu \big[ \int_{\Omega} H(x,\theta) N^T N d\Omega \big] \approx M_0 + \sum_{r=1}^{q} M_r \xi_r(\theta)
\]

\( (3) \)

with:

\[
M_0 = \mu \big[ \int_{\Omega} \mu(x) N^T N d\Omega \big] \quad M_r = \mu \big[ \int_{\Omega} \sqrt{\lambda_r} f_r(x) N^T N d\Omega \big]
\]

\( (4) \)

In the same way, the stochastic stiffness matrix is written as:

\[
K(\theta) \approx K_0 + \sum_{r=1}^{q} K_r \xi_r(\theta)
\]

\( (5) \)

with:

\[
K_0 = \mu \big[ \int_{\Omega} \mu(x) B^T B d\Omega \big] \quad K_r = \mu \big[ \int_{\Omega} \sqrt{\lambda_r} f_r(x) B^T B d\Omega \big]
\]

\( (6) \)
D is the matrix of elastic coefficients.
When one attribute to zones or sub-structures a level of uncertainty, the stochastic equilibrium equation of the damped non-linear structure submitted to a deterministic harmonic excitation, can be written in the following form:

\[
\left(-\omega^2 M(\theta) + (1 + i\eta) K(\theta)\right) U(\omega, \theta) + f_{nl}(\dot{U}, U) = f_e(\omega)
\]

(7)

Or again:

\[
\left[\mathbf{Z}_0(\omega) + \Delta Z(\omega, \theta)\right] U(\omega, \theta) + f_{nl}(\dot{U}, U) = f_e(\omega)
\]

(8)

with: \(\mathbf{Z}_0(\omega) = \left(-\omega^2 M_0 + (1 + i\eta) K_0\right)\), the mean dynamic stiffness matrix;

\(\Delta Z(\omega, \theta) = \sum_{r=1}^{q} \left(-\omega^2 M_r + (1 + i\eta) K_r\right) \xi_r\), the stochastic dynamic stiffness matrix; \(U(\omega, \theta)\), the response vector of the stochastic model; \(f_{nl}(\dot{U}, U)\), the non-linear forces vector and \(f_e(\omega)\), the Applied forces vector.

The equation (8) can be rewritten in the form:

\[
\mathbf{Z}_0(\omega) U(\omega, \theta) + f_{nl}(\dot{U}, U) = f_\Delta(\omega, \theta) + f_e(\omega)
\]

(9)

\(f_\Delta(\omega, \theta) = -\Delta Z(\omega, \theta) U(\omega, \theta)\) is the random forces vector associated to the unknown modifications of the initial structure. The relation (9) then interpreted as the dynamic equilibrium equation of the non-linear initial model submitted to solicitations \(f_\Delta(\omega, \theta)\).

In the practice, the resolution of the non-linear problem (9), using the Monte Carlo simulation (MC), is very expensive. The condensation of this model by standard reduction method proves to be insufficient in term of predictivity and robustness vis-à-vis of the parametric perturbations. One can propose then to develop a dynamic condensation method adapted to the stochastic models [7] according to iterative procedure.

So, the dynamic response of the perturbed system (9) can be expressed by standard transformation basis \(T_0\) obtained from the mean model, enriched by a static residual \(R\) such that:

\[
U(\omega, \theta) = T_0 c(\omega, \theta) + R f_\Delta(\omega, \theta)
\]

(10)

The reduction basis \(T\) is constructed by exploiting the condensation basis \(T_0\) and the static displacements \(R f_\Delta\) associated to a set of static loads \(F_\Delta\) which are representative of the perturbations \(\Delta Z(\omega, \theta)\):

\[
T = \begin{bmatrix} T_0 & \Delta T \end{bmatrix} \ ; \ \Delta T = R f_\Delta
\]

(11)

On can propose to construct the reduction basis \(T_0\) by the Karhunen-Loeve approach (KL) in the context of a dynamic sub-structuring. This basis presents the same advantages with that of a Ritz basis.

**Equivalent linearization Concept**

The basic idea of this method consists in replacing the differential equation of a non-linear system by an "equivalent" linear differential equation such as the difference between the two systems is minimal [9]. One can adopt the concept of equivalent linearization that consists in expressing the cubic non-linear forces vector in stiffness and damping in the following form:

\[
f_{nl}^{ij}(\omega, \theta) = \delta_{ij}\left(U_i(\omega, \theta) - \dot{U}_i(\omega, \theta)\right)^3 + \gamma_{ij}\left(U_i(\omega, \theta) - \dot{U}_i(\omega, \theta)\right)^3
\]

(12)

where \((\delta_{ij}, \gamma_{ij})\) are the non-linear stiffness and damping coefficients.

One can replace the equation (12) by a combination of linear elements such as:

\[
f_{nl}^{ij}(\omega, \theta) = B_{eq} U_i(\omega, \theta) + K_{eq} U_i(\omega, \theta)
\]

(13)

with: \(B_{eq}(\omega, \theta) = \frac{i}{4} \omega^2 \delta_{ij}\left(U_i^2 + U_j^2 - 2 Re(U_i\dot{U}_j)\right)\) ; \(K_{eq}(\omega, \theta) = \frac{i}{4} \gamma_{ij}\left(U_i^2 + U_j^2 - 2 Re(U_i\dot{U}_j)\right)\).

Consequently, the equation (8) takes the following form:

\[
\left[\mathbf{Z}_0(\omega) + \Delta Z(\omega, \theta) + j\omega R_{eq}(\omega, \theta) + \mathbf{K}_{eq}(\omega, \theta)\right] U(\omega, \theta) = f_e(\omega)
\]

(14)

The obtained stationary solution is based on the projection of the non-linear response on the calculated modal basis by the modal perturbation method and an iterative calculation provided of a convergence criteria.
This criterion exploits the difference between the responses, \( U_k \) and \( U_{k+1} \) obtained respectively at the iteration \( k \) and \( k+1 \). \[
\dot{U}_{k+1} = (1 - \alpha) U_{k+1} + \alpha U_k \quad ; \quad \left\| \dot{U}_{k+1} - U_k \right\| / \left\| U_k \right\| \leq \varepsilon_r
\]
where : \( 0 < \alpha < 1 \) et \( \varepsilon_r \) is the tolerance error fixed a priori \( ( 15 ) \).

3. ROBUST MODAL SYNTHESIS METHOD VIS-A-VIS OF THE STRUCTURAL MODIFICATIONS AND UNCERTAINTIES.

The stochastic model condensed by a reduction basis of the nominal model \( T_\theta \) can be written, in frequency domain, in the following form:

\[
Z^c_0(\omega) U^c(\omega, \theta) + f_{nl}^c(U^c, U^c) = f^c_\Delta(\omega, \theta) + f^c_e(\omega)
\]

\[
Z^c_0(\omega) = T_\theta^T Z_\theta(\omega) T_\theta = -\omega^2 M^c_\theta + (I + i\eta) K^c_\theta
\]
is the mean matrix of the condensed dynamic stiffness; \( f^c_\Delta(\omega, \theta) \) is the condensed vector of the stochastic forces; \( f^c_{nl}(U^c, U^c) \) is the condensed vector of non-linear forces; \( f^c_e(\omega) \), the condensed vector of the Applied forces.

One can propose to construct the reduction basis \( T_\theta \) by exploiting the Karhunen-loeve (KL) in the context of the dynamic sub-structuring.

The optimal modes are extracted of the frequencies responses sampled of each sub-structure submitted to the junctions forces to interfaces. These modes \(( \phi_1, \ldots, \phi_k, \ldots, \phi_M )\) are determined by using the "Snapshots" method in the frequency domain, where one can suppose that \( \phi_k(x) \) \(( k = 1, \ldots, M)\) is a linear combination of samples:

\[
\phi_k = \sum_{i=1}^{M} \alpha_{i,k} U_i
\]

where: \( \alpha_{i,k} \) are the eigenvectors of the problem: \( C A_k = \lambda_k A_k \) with \( k = 1, \ldots, M \); \( C \) is the covariance matrix with a general term: \( C_{ij} = \frac{1}{M} U_i^T U_j \) \(( i, j = 1, \ldots, M) \) and \( A_k = [ \alpha_{1,k} \ldots \alpha_{M,k} ]^T \). The responses \( U_i \) \(( i = 1, \ldots, M) \) intervening in \( (17) \) are obtained from the movement equation of one sub-structure submitted to the only junctions forces \( F_j \):

\[
( K_\theta - \omega^2 M_\theta ) U(\omega) = F_j(\omega)
\]

These forces being a priori unknown, the proposed method in [10] uses the technique "Single Composite Input" to excite the sub-structure at the level of the j DOF of junctions.

Thus, at each frequency \( \omega_k \), one can introduce an excitation force \( F_j(\omega) = [ F_j(\omega_k) 0 ]^T \) in the form:

\[
F_j(\omega_k) = \begin{bmatrix}
\frac{1}{i\omega_k - i\Omega_1} & \frac{1}{i\omega_k - i\Omega_2} & \cdots & \frac{1}{i\omega_k - i\Omega_j}
\end{bmatrix}^T \text{ with } i^2 = -1
\]

Frequencies \( \Omega_k \) \(( k=1, 2, \ldots, j)\) are chosen arbitrarily in the sampling range \([0, \Omega]\).

The modes \( \phi_k \) are used as a condensation basis and the transformation KL takes the following form:

\[
T_{KL} = [ \phi_1 \ldots \phi_M ] \quad k \leq M
\]

The enrichment of the basis \( T_{KL} \) by the random static residual uses the M responses \( U_{\theta_1}, \ldots, U_{\theta_M} \). For each pulsation \( \omega_k \) \(( k=1, \ldots, M)\), the force vector of the uncertain parametric modification is written as:

\[
f^c_\Delta = -\left( \Delta K - \omega_k^2 \Delta M \right) U_\theta(\omega_k)
\]
The set of vectors \( f_\Delta \) allows the generation of the force basis \( F_\Delta \) and the static residual matrix \( R_\Delta = K^{-1} F_\Delta \) associated to the stochastic modification forces permits the construction the enriched optimal basis of KL :

\[
T = \begin{bmatrix} T_{KL} & \mid & R_\Delta \end{bmatrix}
\]

One can dispose finally of condensed stochastic model to calculate the response of the stochastic structure while using the modal perturbation method to reduce the calculation times.

4. CALCULATION OF THE RANDOM RESPONSES BY THE MODAL PERTURBATION METHOD

To solve the problem of bad predictability of the stochastic responses around the resonances, one can use the modal perturbation method.

The generalized problem \( K_\theta Y_j = \omega_j^2 M_\theta Y_j \) is firstly solved in view of identifying the eigenfrequencies \( \omega_j \) of the mean structure and their associated eigenvectors \( Y_j \) \( (j=1,\ldots,N) \).

To evaluate the random response \( U(\omega,\theta) \) of the non-linear model, one can use the truncated random eigenmodes \( (N' \ll N) \):

\[
U(\omega,\theta) = \sum_{j=1}^{N} \alpha_j(\omega,\theta) Y_j(\theta)
\]

The normal coordinates \( \alpha_j(\omega,\theta) \) are expressed by :

\[
\alpha_j(\omega,\theta) = \frac{\left( Y_j(\theta) \right)^T f_\theta(\omega)}{-\omega^2 + (1+i\eta)(\omega_j(\omega))^2 + i\omega B_j(\omega) + K_j(\omega)}
\]

where :

\[
\widetilde{B}_j(\omega) = \left( Y_j(\theta) \right)^T B_{eq}(\theta) Y_j(\theta) \quad ; \quad \widetilde{K}_j(\omega) = \left( Y_j(\theta) \right)^T K_{eq}(\theta) Y_j(\theta)
\]

The first order decomposition of the random eigenmodes, can be written as :

\[
\omega_j(\theta) = \omega_j + \sum_{r=1}^{q} \frac{\partial \omega_j}{\partial \xi_r} \xi_r(\theta) \quad ; \quad Y_j(\theta) = Y_j + \sum_{r=1}^{q} \frac{\partial Y_j}{\partial \xi_r} \xi_r(\theta)
\]

The calculation of the first order sensitivities of the eigen problem (sensitivities of the eigenfrequencies and the eigenmodes), can be effectuated by using the works proposed by Adelman and al [4] that exploit the classic approaches of Fox and Kapoors and of Nelson [5]. The eigenvalues sensitivity is given by:

\[
\frac{\partial \lambda_j}{\partial \xi_r} = Y_j^T \left( \frac{\partial K}{\partial \xi_r} - \lambda_j \frac{\partial M}{\partial \xi_r} \right) Y_j \quad \text{and} \quad \frac{\partial \omega_j}{\partial \xi_r} = \left( 1/2 \omega_j \right) \left( \partial \lambda_j / \partial \xi_r \right)
\]

The eigenvectors sensitivity is given by :

\[
\frac{\partial Y_j}{\partial \xi_r} = V + c Y_j
\]

Where \( V \) is the solution of the modified system obtained by applying a penalization method to the \( k^{th} \) row and column de \( \left( K_\theta = \lambda_j M_\theta \right) \) such that \( V_k = 0 \) :

\[
\left( K_\theta = \lambda_j M_\theta \right)_{\text{mod}} . V = - \left( \frac{\partial K}{\partial \xi_r} - \lambda_j \frac{\partial M}{\partial \xi_r} \right) Y_j
\]

The expression of \( c \) is given by the relation:

\[
c = - \left( \frac{1}{2} Y_j^T \frac{\partial M}{\partial \xi_r} Y_j + Y_j^T M_\theta V \right)
\]

The derivative of the mass and stiffness matrices are respectively expressed by:

\[
\frac{\partial M}{\partial \xi_r} = \int_{\Omega} \sum_{i} \sqrt{\lambda_i} f_i(x) N^T \partial \Omega e \quad ; \quad \frac{\partial K}{\partial \xi_r} = \int_{\Omega} \sum_{i} \sqrt{\lambda_i} f_i(x) B^T DB \partial \Omega e
\]

A similar procedure is applied when the condensed model is used.
5. NUMERICAL SIMULATION

The clamped-clamped structure is constituted of two sub-structures assembly of plates (figure 1). The FEM includes 3132 DOF distributed in 792 internal DOF of SS1, 1944 internal DOF of SS2 and 198 junctions DOF for each sub-structure. The geometric and mechanical characteristics are: \( h=1 \times 10^{-3} \text{m} \); \( E_0 = 2,1 \times 10^{11} \text{N/m}^2 \); \( \rho_0 = 7800 \text{kg/m}^3 \). Along the two sub-structure interface, one can consider a line of 33 localized stiffness non-linearities at the junctions DOF, of values: \( k = 10^5 \text{N/m} \); \( \gamma = 10^{22} \text{N/m}^3 \).

The dynamic analysis is realized in the frequency band \([0 – 220 \text{Hz}]\) containing the first 26 modes. By the Karhunen-Loève method, one can consider 300 samples with 220 modes for SS1 and 260 modes for SS2 in the useful frequency band. The initial model is then reduced to a condensed model of 480 DOF.

One can examine firstly the effect of the different localized non-linearities at the junctions of the structure. In the second step, one can examine the behavior of the structure submitted to high uncertain parametric modifications. The table 1 illustrates the errors on the frequencies and the eigenvectors where one can compare the first 28 eigenmodes calculated from the condensed model to those of the reference model (without modification). One can note that the condensation method of KL type permits to obtain the first 28 eigenmodes of the model without modification with a good precision.

![Figure 1: FEM of PLATES](image)

<table>
<thead>
<tr>
<th>Deterministic eigenfrequency (Hz)</th>
<th>3132 DOF</th>
<th>480 DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.65</td>
<td>7 e-008</td>
</tr>
<tr>
<td>2</td>
<td>12.96</td>
<td>6 e-007</td>
</tr>
<tr>
<td>3</td>
<td>24.79</td>
<td>1 e-06</td>
</tr>
<tr>
<td>4</td>
<td>33.32</td>
<td>2 e-06</td>
</tr>
<tr>
<td>5</td>
<td>35.11</td>
<td>1 e-06</td>
</tr>
<tr>
<td>6</td>
<td>47.42</td>
<td>7 e-007</td>
</tr>
<tr>
<td>7</td>
<td>57.04</td>
<td>6 e-006</td>
</tr>
<tr>
<td>8</td>
<td>61.66</td>
<td>3 e-008</td>
</tr>
<tr>
<td>9</td>
<td>70.49</td>
<td>1 e-06</td>
</tr>
<tr>
<td>10</td>
<td>79.36</td>
<td>1 e-007</td>
</tr>
<tr>
<td>11</td>
<td>88.46</td>
<td>4 e-008</td>
</tr>
<tr>
<td>12</td>
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<td>6 e-006</td>
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<tr>
<td>13</td>
<td>102.41</td>
<td>1 e-006</td>
</tr>
<tr>
<td>14</td>
<td>116.05</td>
<td>1 e-006</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Deterministic eigenfrequency (Hz)</th>
<th>3132 DOF</th>
<th>480 DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>127.18</td>
<td>1 e-006</td>
</tr>
<tr>
<td>16</td>
<td>141.57</td>
<td>4 e-007</td>
</tr>
<tr>
<td>17</td>
<td>144.39</td>
<td>6 e-007</td>
</tr>
<tr>
<td>18</td>
<td>153.53</td>
<td>1 e-006</td>
</tr>
<tr>
<td>19</td>
<td>158.44</td>
<td>8 e-007</td>
</tr>
<tr>
<td>20</td>
<td>161.41</td>
<td>4 e-007</td>
</tr>
<tr>
<td>21</td>
<td>169.56</td>
<td>3 e-007</td>
</tr>
<tr>
<td>22</td>
<td>180.49</td>
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<tr>
<td>23</td>
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<td>24</td>
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<td>25</td>
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<td>2 e-006</td>
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<tr>
<td>26</td>
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<tr>
<td>27</td>
<td>222.98</td>
<td>1 e-006</td>
</tr>
<tr>
<td>28</td>
<td>230.88</td>
<td>2 e-005</td>
</tr>
</tbody>
</table>
The figure 2 represents the non-linear response of the reference and condensed model (without modification) at the position « N0 ». A very good coincidence between the two responses is illustrated by the curve of difference in displacement term on the same figure. The effect of localized non-linearities is visualized on the figure 2, as well as on the Nyquist diagram (figure 3).

Figure 2: Non-linear frequency responses at the position « N0 », reference and condensed model (without modification)

Figure 3: Nyquist Diagram

One can consider the case (figure 1) where the structure is submitted to three types of uncertain parametric modifications (modulus of elasticity, thickness and density).

Uncertainties on the thickness h are introduced through a decoupling of the membrane effects ($K_m$) and of the bending ($K_f$) in the stiffness matrix: $K_{zone} = h K_{zone}^m + h^3 K_{zone}^f$ (table 2).

Table 2: Uncertain Parametric Modifications

<table>
<thead>
<tr>
<th>Modifications per zones</th>
<th>Sub-structure 1</th>
<th>Sub-structure 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Young's Modulus</td>
<td>Thickness</td>
</tr>
<tr>
<td></td>
<td>X 100</td>
<td>X 5</td>
</tr>
<tr>
<td>uncertainties</td>
<td>10 %</td>
<td>10 %</td>
</tr>
</tbody>
</table>

One can present on figure 4 the distance in form between the first eigenvectors of the initial model and the perturbed model.

The enrichment of the transformation basis (KLE) can be effectuated by using 120 static residues (40 residues for SS1 and 80 residues for SS2).

The uncertain eigenvectors calculated from the perturbed reduced models are reconstituted by using the respective reduction basis then compared to the uncertain eigenvectors obtained from the reference model. One can illustrate the distance in form between these vectors for the KL and KLE method.

The table 3 and the figure 5 present respectively the relative errors on the eigenfrequencies and the distance in form between the reconstituted uncertain eigenvectors obtained using the KL and KLE methods and the exact uncertain eigenvectors.

The figure 6 shows that one can obtain a good correspondence between the curves of the mean and the extreme statistics of the non-linear uncertain response calculated by the modal perturbation method (MP) for 1000 samples, of the reference modified model and the condensed modified model.

To highlight the effect of the reduction of the non-linear model, one can specify the CPU time of the reference non-linear modified model (CPU_REF = 140 H) and the CPU time of the condensed non-linear model (CPU_KLE = 35 H).
### Table 3: Precision of eigenfrequencies ($f_e$) and eigenvectors ($U_e$). Modified model with uncertainties

<table>
<thead>
<tr>
<th>Random eigenfrequency (Hz) (first moment)</th>
<th>KL</th>
<th>KLE</th>
<th>KL</th>
<th>KLE</th>
</tr>
</thead>
<tbody>
<tr>
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**Figure 4:** Distance in form between the exact perturbed structure and the initial structure.

**Figure 5:** Distance in form between the exact perturbed structure and the condensed perturbed structure by the classic KL method (at left) and by the enriched KLE method.
6. CONCLUSIONS

In this article one can propose a new modal synthesis method of the linear and non-linear structures. This method that consists in coupling the Stochastic Finite Element method and a robust model condensation method vis-à-vis of the structural modifications and uncertainties. This condensation method is based on the Karhunen-loeve approach.

The analysis of the simulation results shows that this method constitutes an interesting alternative to the classic reduction methods that are maladjusted to the condensation of the Stochastic Finite Element Models and to the non-linear structures.

Otherwise, the comparison of the calculation costs permitted to highlight the performances of the double condensation by the enriched Karhunen-Loeve method followed of the modal perturbation method.

The presented method permits, by the reduction of the model size, the reduction of the calculation time and its predictivity, to response to needs of reanalysis met in the optimization iterative procedures or in the dynamic analysis of the non-linear problems of complex structures.

REFERENCES