Identification of Structural Parameters Based on Inverse Modification Theory

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ABSTRACT

During the service life of a structure its stiffness, mass and damping parameters may be changed due to design modifications or due to a damage event. The effects of such modifications are usually analysed by either supplementing the initial Finite Element model with a model of the modification or, more directly, by calculating the modified structure response using structural modification theory. In practice the parameters of the modification are often unknown, for example when the modification was caused by a crack or by bolting a stiffener to the structure. In such cases a model updating approach can be applied where the parameters of the modified structure model are fitted to the test data of the modified structure. The drawback of this approach comes from the necessity to include not only the additional parameters related to the modification but also parameters related to the unmodified part of the structure since the model of the unmodified structure might include not only parametric errors but also physical modelling errors (e.g. due to oversimplification). Therefore, the model of the unmodified structure is usually fitted to test data of the unmodified structure in advance. In any case the success of the model updating approach is based on the ability of the initial model to represent the physics, i.e. the model should not contain physical model errors but only errors in the parameters of the model [1].

In the present paper an approach based on inverse modification theory is presented which allows the identification of the modification parameters. This approach includes the more classical model updating approach where the parameters of an initial analytical are fitted to measured frequency responses. However, the procedure also allows to use only measured frequency responses of the initial unmodified structure instead of analytical responses, i.e. in this case there is no need for a model of the whole structure, only a model of the modification is necessary.

Structural modification theory is used as a tool to calculate the frequency response of a modified structure given the frequency response of the unmodified structure and the location, the type and the magnitude of the modification [4]-[7]. This theory is used here to solve the inverse problem: identify the magnitude of the modification with the frequency responses of the modified and unmodified structure taken either from analysis or test.

1. BASICS OF COMPUTATIONAL MODEL UPDATING

Computational updating procedures are aimed at fitting selected model parameters such that the test/analysis deviations are minimised. Using appropriate residuals (containing the test/analysis differences) the following objective function J can be derived:

\[ J(p) = \Delta z^T W \Delta z + p^T W_p p \rightarrow \min \]  

(1)

with: \( \Delta z \) residual vector containing the test/analysis differences, \( W, W_p \) weighting matrices.
The minimisation of equation (1) yields the desired correction parameters $p = [p_i]$, $i=1,2…p$ no. of correction parameters. The second term in equation (1) is used to constrain the parameter variation. The residuals $\Delta z = z_T - z(p)$ ($z_T$: test data vector, $z(p)$: corresponding analytical data vector) usually depend in a non-linear way on the parameters. Thus the minimisation problem is also non-linear and must be solved iteratively. One way is to use the classical sensitivity approach (e.g. references [1] – [3]) where the analytical data vector is linearized by a Taylor series expansion truncated after the first term which leads to:

$$
\Delta z = \Delta z_0 - G_0 \Delta p
$$

(2)

with:

- $p_0$ design parameter vector at linearization point (index " 0"),
- $\Delta p = p - p_0$ design parameter changes,
- $r_0 = z_T - z(p_0)$ test/analysis differences (residual vector) at linearization point, for example, the differences between eigenfrequencies and/or mode shapes and/or frequency responses
- $G_0 = \partial z(p) / \partial p |_{p=p_0}$ sensitivity matrix at linearization point.

Introducing eq.(2) into eq.(1) and constraining only the parameter changes yields the linear problem (3) which has to be solved for the parameter changes $\Delta p$ in each iteration step for the actual linearisation point.

$$(G_0^T W G_0 + W p) \Delta p = G_0^T W r_0 \quad (3)$$

For $W p = 0$ equation (3) represents a standard weighted least squares problem.

The above minimisation procedure requires an appropriate parameterisation of the model. The most widely used parameterisation of the finite element (FE) model matrices concerning physical parameters like mass, material or beam cross section parameters is given by:

$$
K = K_A + \sum \alpha_i K_i, \quad i = 1…n_a
$$

(4a)

$$
M = M_A + \sum \beta_j M_j, \quad j = 1…n_b
$$

(4b)

$$
D = D_A + \sum \gamma_k D_k, \quad k = 1…n_c
$$

(4c)

with:

- $K_i$, $M_j$, $D_k$ initial analytical stiffness, mass and damping matrices,
- $p = [\alpha, \beta, \gamma]$ vector of unknown modification parameters,
- $K_i$, $M_j$, $D_k$ given substructure matrices defining location and type of parameter uncertainties.

This parameterisation allows to modify the mass, stiffness and damping parameters of selected substructures.

The solution of eq.(3) yields the parameter changes $\Delta p$ which are used to update the parameter vector in the next iteration step by

$$
p(p_0 + \Delta p) = p(p_0) + \Delta p \quad (5)
$$

3. CALCULATION OF RESIDENTIAL VECTOR AND RESPONSE SENSITIVITIES

In this paper we report about minimizing the residual vector

$$
r = u^m - u(p_i)
$$

(6)

which represents the difference of frequency responses $u^m$ of a modified structure due to an excitation force vector $F$ applied at one or more of the $m$ modification DOFs and the frequency responses $u(p_i)$ of an unmodified structure to identify the unknown modification parameters $p_i$.

The standard procedure to calculate response sensitivities is based on deriving the equation of motion,

$$
K - \omega^2 M + j \omega D = F, \quad \text{after introduction of eqs.}(4), \text{with respect to the parameter vector } p \text{ which yields}
$$
sensitivity matrix at linearization point "0" where
\[
H_{nm} \quad (n,m)- \text{FRF } -\text{matrix}
\]
\[
\begin{align*}
\frac{\partial u(p_i)}{\partial p_i} & = -H_{ui} \left[ \frac{\partial u(p_i)}{\partial p_i} \right]_{p_i=p_i0} \\
\frac{\partial u(p_i)}{\partial p_i} & = \left[ -H_{ui} \left[ \frac{\partial u(p_i)}{\partial p_i} \right] \right]_{p_i=p_i0} F_i
\end{align*}
\]

Although this procedure seems to be a straightforward approach it has turned out to fail in many practical applications. The main reasons are:

1. The sensitivities which are calculated from a first order Taylor series expansion of the response are not accurate enough in the vicinity of the resonance peaks to determine the search direction.
2. If the FRF matrices are calculated in the standard way by modal superposition the modal truncation error affects the accuracy of the sensitivity matrix considerably.
3. It is difficult to estimate a physically realistic damping modification matrix, $D$. 

### 4. Calculating the Modified Structure Response Based on Modification Theory

In the present chapter we introduce an alternate way of calculating the sensitivity matrix which is based on exact structural modification formulas. The derivation of the structural modification formula uses the definition of the FRF- matrix, $H$, as the inverse of the dynamic stiffness matrix, $Z$. If partitioned with respect to modification DOFs, called the "m"-set and unmodified DOFs, called the "u"-set, where no structural modifications are applied the identity

\[
\begin{bmatrix}
Z_{uu} & Z_{um} \\
Z_{mu} & Z_{mm}
\end{bmatrix}
\begin{bmatrix}
H_{uu} & H_{um} \\
H_{mu} & H_{mm}
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

yields the following relations between the submatrices:

\[
H_{mn} = (Z_{mn} - Z_{mn}^{-1} Z_{um}^{-1} Z_{mn})^{-1} = Z_{mn,c}^{-1} \]  

\[
H_{mn} = Z_{mn}^{-1} Z_{mn} H_{mn}
\]

If the structure is modified at the "m"-DOFs by a dynamic stiffness matrix
\[
\Delta Z = -\omega^2 \Delta M + j\omega \Delta D + \Delta K
\]

but is kept unmodified at the "u"-DOFs the dynamic stiffness submatrices of the modified structure (denoted with an upper index m) are given by:

\[
Z_m^m = Z_m^m + \Delta Z; \quad Z_{um}^m = Z_{um} \quad \text{and} \quad Z_{um}^m = Z_{um}
\]

Eqs. (8b,c) also hold for the modified structure taking eqs. (10) into account which results in
\[ \mathbf{H}_{mn}^\Delta = (\Delta \mathbf{Z} + \mathbf{Z}_{mn}^{-1} \mathbf{Z}_{mm} \mathbf{Z}_{mn} \Delta ^{-1})^{-1} (\Delta \mathbf{Z} + \mathbf{Z}_{mm} \Delta ^{-1})^{-1} = (\mathbf{Z}_{mm}^{-1} \mathbf{I} + \mathbf{Z}_{mm}^{-1} \mathbf{Z}_{mm} \mathbf{Z}_{mn} \Delta ^{-1})^{-1} \Rightarrow \]

\[ \mathbf{H}_{mn}^\Delta = (\mathbf{I} + \mathbf{H}_{mm} \Delta ^{-1})^{-1} \mathbf{H}_{mn} = \mathbf{I} \mathbf{H}_{mn}^{-1} \]  

(11a)

where \( \mathbf{I} = \mathbf{I} + \mathbf{H}_{mm} \Delta \)  

(11b)

The FRF- submatrix related to the unmodified DOFs is obtained from

\[ \mathbf{H}_{mn}^\Delta = -\mathbf{Z}_{mm}^{-1} \mathbf{Z}_{mn} \mathbf{H}_{mn}^\Delta = -\mathbf{Z}_{mm}^{-1} \mathbf{Z}_{mn} \mathbf{I} \mathbf{H}_{mm}^{-1} \Rightarrow \]

\[ \mathbf{H}_{mn}^\Delta = \mathbf{H}_{mn} \mathbf{I} \mathbf{H}_{mm}^{-1} . \]  

(11c)

The submatrices of the modified structure can thus be expressed in terms of the FRF-submatrices of the unmodified structure and of the dynamic stiffness matrix \( \mathbf{Z} \) of the modification (these formulas were also derived by an alternate formulation in [6]).

5. FORMULATION OF THE INVERSE PROBLEM

We are interested to solve the inverse problem: calculation of the modification matrix \( \mathbf{Z} \) when the FRF-matrices of the unmodified and the modified structures are given either by measurement or by analysis. We use first eqs.(4) to parameterize the modification matrix by

\[ \Delta \mathbf{Z} = \sum \mathbf{Z}_i \mathbf{p}_i \]  

(12)

as already defined in eq.(7). Next we want to calculate the modification parameters by minimizing the difference between the responses of the modified and the unmodified structure excited by a force vector \( \mathbf{F}_e \) applied at one or more of the “m” modification DOFs as defined in eq.(6). \( \mathbf{r}(\mathbf{p}_i) = \mathbf{u}^m - \mathbf{u}(\mathbf{p}_i) \). For a given parameter estimate \( \mathbf{p}_i \) \((i=1,2...\text{no. of parameters})\) the response \( \mathbf{u}(\mathbf{p}_i) \) can be calculated from the unmodified structure response using the FRF-matrices of eqs.(11) which yields the residual

\[ \mathbf{r}(\mathbf{p}_i) = \begin{bmatrix} \mathbf{H}_{mn}^\Delta & \mathbf{H}_{mn} \mathbf{I}^{-1} & \mathbf{F}_e \\ \mathbf{u}^m & \mathbf{u}(\mathbf{p}_i) & \mathbf{F}_e \\ \end{bmatrix} \]  

(13)

where \( \mathbf{I} = \mathbf{I} + \mathbf{H}_{mm} \Delta \mathbf{Z} = \mathbf{I} + \mathbf{H}_{mm} \sum \mathbf{Z}_i \mathbf{p}_i \).

The minimization of the norm of this residual vector with respect to the modification parameters leads to a nonlinear optimization problem which can be solved by a Gauss-Newton approach as follows. Replacing \( \mathbf{u}(\mathbf{p}_i) \) by its first order Taylor series expansion yields the linearized residual at the linearization point denoted with the subscript “0”

\[ \mathbf{r}_o = \mathbf{u}^m - \mathbf{u}_o - \mathbf{G}_o \Delta \mathbf{p} \]  

(14a)

where
Minimizing the objective function \( r^T r \) with respect to \( p \) yields (after appropriate weighting) the parameter changes in a least squares sense as in eq.(3). However, the sensitivity matrix looks different from that of eq.(7). The differentiation of \( I^\Delta \) with respect to the parameters yields \( \partial I^\Delta / \partial p_{i} = -I^\Delta (\partial I^\Delta / \partial p_{i}) I^\Delta = -I^\Delta H_{nn} Z_i I^\Delta \) so that the sensitivity in eq.(14c) can be expressed by

\[
G_{ii} = \begin{bmatrix} H_{nn} I^\Delta Z_i H_{nn} \nabla I^T \\
I^\Delta Z_i H_{nn} Z_i I^\Delta \end{bmatrix} (i=1, \ldots, n_p)
\]  

(14d)

This expression becomes identical to that of eq.(7) in case of \( I^\Delta \approx \mathbf{I} \) which only holds if \( H_{nn} \Delta Z \ll \mathbf{I} \) as is obvious from eq.(11b). Since this assumption is heavily violated around resonances of the unmodified structure it becomes clear why the frequencies in the vicinity of the resonances should not be used with the sensitivity matrix of eq.(7). In the following examples we will show the beneficial effect of using the improved sensitivity formulation of eq. (14d).

Using the FRF modification formulas (11) has the advantage that during the iteration there is no reanalysis necessary with the updated finite element model. The finite element model is only used at the beginning of the iteration to calculate the initial FRF-matrix estimate. It must be noted that it is not advisable to calculate the FRF-matrices as usual by truncated modal superposition but rather by direct frequency-wise solution of the equation of motion. The reason is that the inverted FRF matrices are extremely sensitive to truncation errors.

It would even be possible to identify the modification parameters without utilizing any finite element model and start the iteration from experimental FRF matrices measured from the unmodified structure. In this case the sensitivity matrix (14d) is calculated from the initial experimental FRF-data updated during the iteration by the modification formulas. However, such an approach is likely to introduce the difficulty of unbiased estimates arising when measured matrices have to be (pseudo-) inverted like in eqs.(3) and (11), see e.g. ref[7].

6. ILLUSTRATIVE EXAMPLES

6.1 Identifying the modification parameters of a 5-DOF spring-mass system using simulated test data

To check the identification procedure we used the 5-DOF spring system shown in fig.1. The following numerical data for the spring stiffnesses were used to calculate the responses of the unmodified system:

- Stiffness parameters [N/m] :
  - \( K_1 = K_2 = K_4 = K_5 = 27216 \) ;  \( K_3 = 39191 \) ;  \( K_6 = K_7 = K_8 = K_9 = 17901 \) ;  \( K_{10} = K_{11} = 15312 \)

- Tip masses [kg] :
  - \( M_{1-5} = [0.429 0.574 0.655 0.674 0.729] \)

- Mass matrices \( M_{ij} \) coupling DOFs i-j, i=1-2, 2-3, 3-4 and 4-5:
  - \( M_{ij} = m_c \begin{bmatrix} 1 & 0.5 \\
0.5 & 1 \end{bmatrix} \) with \( m_c = 0.025 \) kg ;

- Mass coupling was also assumed for DOFs 1-3 and 3-5 with \( m_c = 0.315 \) kg.

- Modal damping:
  - \( \zeta = [1.17 \ 0.94 \ 0.92 \ 0.91 \ 0.74] \% \)

To calculate the response residual the exciter force vector \( F_e = [0 \ 1 \ -1 \ 0 \ 1] \) N was applied.
The calculated results for the undamped unmodified model are:
Eigenfrequencies Hz:
\[ f_{\text{unmod}} = [31.82 \; 41.49 \; 54.89 \; 59.71 \; 66.09] \ \text{Hz} \]

The FRFs were calculated using all five mode shapes (no truncation error). Two stiffness modifications were introduced:
Modification 1: the spring stiffness no.7 between DOFs 2 and 3 was increased by 50%.
Modification 2: the spring stiffness no.11 between DOFs 3 and 5 was decreased by -50%.

The calculated eigenfrequencies for the undamped modified model are:
\[ f_{\text{mod}} = [31.67 \; 40.03 \; 54.21 \; 61.34 \; 67.37] \ \text{Hz} \]

The opposite sign of the modification was introduced to check if the sensitivity formulation was suited to distinguish the different search directions. Therefore, the relatively large stiffness modifications has only a relatively small effect on the eigenfrequency differences.

Fig.2 shows the convergence characteristics of the minimization process using the classical response sensitivity of eq.(7) compared to the improved sensitivity formulation of eq.(14d). Five excitation frequencies around the 5 resonance peaks were used to calculate the residual vector at all 5 DOFs. Fig.2a clearly shows the superior convergence of the formulation eq.(14d) converging monotonically to the nominal values, 0.5 and -0.5, whereas the classical formulation tends to an oscillatory behaviour with the objective function not yet minimized to zero after 20 iteration steps as shown in fig.2b. It must be mentioned that the result of this exercise only shows that the procedure was correctly implemented since we have used the same type and the same location for the modification matrices as for those used to simulate the test data (consistent assumptions).
6.2 Identifying the modification parameters of a beam with a bolted joint using real test data

This second example was used to study the identification process using real test data of a structure consisting of two aluminium beams connected by a bolted joint as shown in fig.3. In order to model the joint stiffness a special 3-DOF finite element was derived allowing to model the rotational stiffness between two beams connected by a hinge and a rotational spring as shown in Fig.4. This model is particularly useful to distribute the rotational stiffness to translational DOFs.

Beam properties: Area: \(A = 2.24 \times 10^{-4} \text{ m}^2\), area moment of inertia: \(I = 1.438 \times 10^{-8} \text{ m}^4\), Young's modulus: \(E = 7 \times 10^{10} \text{ N/m}^2\), mass density: 2650 kg/m\(^3\), joint mass distributed to DOFs 2-4: 0.0525 kg, sensor and cable masses: 0.1 kg (distributed over length), mass contribution of soft support springs: 0.05 kg, stiffness of support springs: 400 N/m.

Fig. 3 Laboratory test beam with a bolted joint

Fig. 4 Rotational spring element, stiffness \(k_d\), connecting two rigid beams

The force – displacement relation of the element restrained at DOFs 1 and 2 is calculated from \(f_j = k_{i}u_{i}\) where \(k_{i} = 4k_{d}/L^2\) represents the translational stiffness of the element. The equilibrium conditions result in the force relations

\[
\begin{bmatrix}
 f_1 \\
 f_2 \\
 f_3
\end{bmatrix} =
\begin{bmatrix}
 1 \\
 -2 \\
 1
\end{bmatrix} \begin{bmatrix}
 u_1 \\
 u_2 \\
 u_3
\end{bmatrix} = \begin{bmatrix}
 k_{1} \\
 -2k_{2} \\
 k_{3}
\end{bmatrix}
\]

From the principal of virtual work it follows that the corresponding displacements transform in a contragredient manner by \(u_j = T^j[u_1 \ u_2 \ u_3]^T\) which, after substitution into the above relation, yields the force-displacement relation \(f = k_{s}u\) of the free/free element where \(k_{s}\) is the stiffness matrix:

\[
k_{s} = k_{i} \begin{bmatrix}
 1 & -2 & 1 \\
 -2 & 4 & -2 \\
 1 & -2 & 1
\end{bmatrix}
\]

This element was assembled to the unmodified beam model (DOFs 2-4 in fig.3) thus introducing the unknown stiffness parameter, \(k_{d}\), to be identified from measured FRF data as described above. Of course, there was also a modification of the mass matrix caused by the joint. This was taken into account by distributing the weighed joint mass of 0.0525 kg to the three joint DOFs.

The result of identifying the joint rotational stiffness is depicted in figs. 5 and 6. Fig.5a shows the superior convergence behavior the improved sensitivity formulation of eq.(14d) versus the usual formulation of eq.(7). The results were obtained by minimizing the response residuals at measured DOFs 1,3 and 5 at 274 frequency points located around three resonances and two antiresonances. Best results were obtained when the
displacement was replaced by the velocity in the response residual which is equivalent to weighting the residual by the excitation frequency. The initial estimate of the rotational spring was related to an equivalent rotational stiffness of the unmodified beam, \( k_d = \gamma \frac{EI}{L} \), with \( L \) = length of the joint and \( \gamma = 2 \), a factor used for the initial estimate so that the lateral stiffness of the joint can be expressed by \( k_l = 4\gamma \frac{EI}{L} \).

\[
\begin{align*}
\text{Fig. 5} & \quad \text{Evolution of rotational stiffness parameter } k_d (a) \text{ and objective function (b) over iteration steps using sensitivity formulations eqs. (7) and (14d)} \\
\text{Fig. 6} & \quad \text{Absolute response envelopes over 5 DOFs before and after stiffness identification}
\end{align*}
\]

The objective function was not completely minimized to zero because the assumed location of the modification did not change the symmetry of the initial model, thus keeping the antisymmetric mode at 165 Hz unchanged in contrast to experimental evidence. In the last iteration step the modification factor \( p \) applied to the element stiffness matrix \( k_e \) of eq.(15) was estimated with \( p = -0.53 \). Fig. 6 compares the absolute response envelopes over 5 DOFs before and after stiffness identification. This figure shows good correlation of the identified and the measures responses. The identified modification stiffness matrix were also used to update the initial finite element model. The eigenfrequencies of the modified model were thus calculated as \( f_{id} = [51.13 \, 165.03 \, 291.88] \text{ Hz} \). The corresponding experimental eigenfrequencies were: \( f_{exp} = [51.25 \, 162.60 \, 287.4] \text{ Hz} \) which means that the percent deviation \( (f_{id}/f_{exp}-1)100 = [-0.23 \, 1.49 \, 1.56] \% \) was considerably improved compared to the deviation \( (f_{ini}/f_{exp}-1)100 = [6.48 \, 1.49 \, 7.54] \% \) corresponding to the initial model frequencies \( f_{ini} = [54.57 \, 165.03 \, 309.08] \text{ Hz} \).

\[
\begin{align*}
\text{Fig. 5} & \quad \text{Evolution of rotational stiffness parameter } k_d (a) \text{ and objective function (b) over iteration steps using sensitivity formulations eqs. (7) and (14d)} \\
\text{Fig. 6} & \quad \text{Absolute response envelopes over 5 DOFs before and after stiffness identification}
\end{align*}
\]

**CONCLUSIONS**

In the present paper we report about using structural modification theory as a tool to identify the parameters of structural modifications. The modifications are parameterized by finite elements assuming that the location and
the type of the modification is known. An improved sensitivity formulation has been derived and its benefits compared to the classical formulation has been demonstrated with two applications, one using simulated test data the other using experimental data of a beam with a bolted joint. Using FRF modification formulas has the advantage that during the iteration no reanalysis based on an updated finite element model is necessary. The finite element model is only used at the beginning of the iteration to calculate the initial FRF-matrix estimate. It would even be possible to identify the modification parameters without utilizing any finite element model and start the iteration from experimental FRF matrices measured from the unmodified structure. In this case the sensitivity matrix is calculated from the initial experimental FRF-data updated during the iteration by the modification formulas. However, such an approach is likely to introduce the difficulty of unbiased estimates arising when measured matrices have to be (pseudo-) inverted. Further investigations on the robustness of the approach in the presence of unavoidable experimental and structural idealisation errors are still necessary to validate the presented approach.

REFERENCES


