COUPLING STOCHASTIC FINITE ELEMENTS – ROBUST CONDENSATION METHODS IN OPTIMIZATION OF STRUCTURES

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NOMENCLATURE AND MAIN SYMBOLS

- **DOF** Degree of freedom
- **MC** Monte Carlo Simulation
- **PC** Polynomial Chaos
- **MEF** Finite Element Model
- **SFEM** Stochastic Finite Element Method
- **$M_\theta$, $K_\theta$** Means mass and stiffness matrices
- **$M(\theta), K(\theta)$** Stochastic mass and stiffness matrices
- **$Z_\theta$, $\Delta Z$** Dynamic stiffness and perturbation matrices
- **$U(\theta)$** Random response vector
- **$T$** Condensation basis
- **$f_\Delta$** Localized stochastic force

ABSTRACT

The consideration of uncertainties in structural dynamics is necessary to improve the models and to increase the robustness of the prediction.

In this article, we propose a strategy coupling the stochastic finite elements methods (SFEM) and a robust condensation method. It is based on a discretization technique of random fields which is established on a Karhunen-Loève development and the use of a dynamic condensation basis enriched by random residual static vectors. These loads are representative of local modifications per zone of the mechanical structure.

The insertion of PC formulation allows to take into account the presence of uncertainties on the design parameters and to analyse the variability of the response in a low costly way, compared to the direct MC simulation. The coupling of the PC with a robust condensation method allows the treatment of the optimization problems which requires simultaneously a good predictivity of the condensed model and a considerable reduction of the calculation costs. At the end of the article, we propose one simulation example which illustrates the performance and the interest of the proposed method.

1. INTRODUCTION

The Prediction of the dynamic behavior of mechanical structures can be improved by considering the influence of uncertainties. These uncertainties are generally linked to model: geometric properties, material characteristics, boundary conditions … They are considered in the models by the parametric approaches using the SFEM method, which combine the classical analysis by finite element and the statistic analysis. Generally, we try to determine the stochastic characteristics of the random responses, from the knowledge of the rand on the conception parameters.

Various techniques exist to solve a such problems [1]. These methods are generally classified in three categories: the MC Simulation method which is often considered as the reference method, the perturbation methods which
are based on a Taylor series expansion of the responses around the means of random variables [2] and the spectral methods which exploit basic functions of the Hilbert space associated to random problems. Generally these functions can be considered as orthogonal polynomials, and particularly a chaos polynomial [3]. In this case, we can use the random variables; the continuous random fields are discretized.

With the aim of obtaining a considerable reduction of the calculation costs and good predictivity of the model, we propose an original method allowing the coupling of the SFEM method and a robust dynamic condensation method towards structural modifications [4] in view to construct a reduced random model. This method makes it possible to attribute to zones or components of the structure a specific level of uncertainties. Furthermore, the robust condensation method towards structural modifications allows the treatment of complex structures with a large FEM.

2. DISCRETIZATION OF RANDOM FIELDS

A random field $H(x,\theta)$ is a collection of random variables indexed by a continuous parameter $x \in \Omega$, where $\Omega$ is the outcome space of $\mathbb{R}^d$, describing the geometry of the system. One procedure of discretization is based on the approximation of $H(\bullet, i)$ by $H(\bullet)$ defined by the means of a finite set of variables $\{x_i, i=1,...,n\}$ which are grouped in a random vector $\chi : \hat{H}(x,\theta) = \mathcal{F}[x,\chi(\theta)]$.

It consists in defining a best approximation compared with some errors estimators, one of which exploits a minimal number of random variables. The most efficacious methods, called Series Expansion Methods, consists in coupling a series development of the random field and a spectral analysis which aims to select the most important terms [3,5,6]. Thus, in the case of homogeneous gaussian field in the form:

$$H(x,\theta) = \mu(x) + \sum_{i=1}^{M} H_i(x) \xi_i(\theta)$$

Where $\{\xi_i(\theta), i=1,...,M\}$ are the gaussian standard normal variables and $\{H_i(x), i=1,...,M\}$ are the deterministic functions.

Let $C(x,x')$ the known covariance function associated to $H(\bullet)$, which is bounded, symmetric and positive definite. The spectral decomposition of $C(x,x')$ can be written as:

$$C(x,x') = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \varphi_i(x')$$

Where $\lambda_i, \varphi_i$ represent respectively the eigenvalues and eigenvectors of $C(x,x')$. The decomposition of Karhunen-Loeve of $H(\bullet)$ on the basis of the eigenfonctions $\varphi_i$ is given by:

$$H(x,\theta) = \mu(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(x)$$

The truncated form:

$$\hat{H}(x,\theta) = \mu(x) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(x)$$

3. GENERAL FORMULATION OF THE STOCHASTIC DYNAMIC MODEL

3.1 Stochastic Finite Element Model

The stochastic mass matrix is obtained using the assembly of the elementary matrices:

$$M = \bigcup_{e} \int_{\Omega_e} N^T S N d\Omega_e$$

With $S(x,\theta) = H(x,\theta)S_q$, $S_q = \rho A$.

Thus, the stochastic mass matrix is written as:

$$M(\theta) = \bigcup_{e} \int_{\Omega_e} N^T S_q H(\hat{x},\theta) N d\Omega_e$$
The replacement of $H(x, \theta)$ by its expression (1), conducts to the following stochastic mass matrix:

$$M(\theta) \approx M_\theta + \sum_{i=1}^{\mathcal{M}} M_i \xi_i(\theta)$$

(6)

With:

$$M_\theta = \bigcup_{\mathcal{e}} \left[ \int_{\Omega} \mu(x) N^T S_\eta N d\Omega_e \right] \quad M_i = \bigcup_{\mathcal{e}} \left[ \int_{\Omega} H_i(x) N^T S_\eta N d\Omega_e \right]$$

(7)

In the same way, the stochastic stiffness matrix is given by:

$$K(\theta) \approx K_\theta + \sum_{i=1}^{\mathcal{M}} K_i \xi_i(\theta)$$

(8)

With:

$$K_\theta = \bigcup_{\mathcal{e}} \left[ \int_{\Omega} \mu(x) B^T D_\theta B d\Omega_e \right] \quad K_i = \bigcup_{\mathcal{e}} \left[ \int_{\Omega} H_i(x) B^T D_\theta B d\Omega_e \right]$$

(9)

$D_\theta$ is the deterministic matrix of elastic coefficients.

3.2 Stochastic dynamic model per zone

When we attribute to zones of the structure a level of uncertainties (only a few parameters are uncertain), the stochastic equilibrium equation of the structure submitted to a harmonic excitation, is written in the following form:

$$\sum_{q=1}^{\mathcal{Q}} \omega^2 M + j \omega B + K \xi_q = f(\omega)$$

(10)

Or:

$$[Z_\theta(\omega) + \Delta Z(\omega, \theta)] U(\omega, \theta) = f_e(\omega)$$

(11)

With: $\Delta Z(\omega, \theta) = \sum_{q=1}^{\mathcal{Q}} (-\omega^2 M_q + j \omega B_q + K_q) \xi_q$ the stochastic dynamic stiffness matrix; $U(\omega, \theta)$, the stochastic response vector of the model and $f_e(\omega)$, the applied harmonic force vector.

The equation (11) can be rewritten in the following form:

$$Z_\theta(\omega) U(\omega, \theta) + f_\Delta(\omega, \theta) = f_e(\omega)$$

(12)

$f_\Delta(\omega, \theta) = \Delta Z(\omega, \theta) U(\omega, \theta)$ is interpreted as the stochastic forces vectors associated to the modifications of the initial structure. In the practical, the resolution of problem (12), using the MC Simulation is very costly. We propose then to exploit a condensation method of this stochastic model.

4. ROBUST DYNAMIC CONDENSATION TOWARDS THE UNCERTAINTIES

The stochastic model is condensed by using a basis $T_\theta$ in the frequency domain, is written is the following form:

$$Z^c_\theta(\omega) U^c(\omega, \theta) + f^c_\Delta(\omega, \theta) = f^c_e(\omega)$$

(13)

$Z^c_\theta(\omega) = \left[ -\omega^2 M^c_\theta + j \omega B^c_\theta + K^c_\theta \right]$ is the condensed mean dynamic stiffness matrix;

$f^c_\Delta(\omega, \theta) = \Delta Z^c(\omega, \theta) U^c(\omega, \theta)$ is the condensed stochastic forces vectors;

$f^c_e(\omega)$, is the condensed vector of the applied forces.

The transformation matrix used in this study [7] constitutes an extension of the return procedure in physical coordinate for different condensation basis of models. To simplify the presentation and without restricting the
generality of the method, we will be limited to the blocked or free interfaces. For these two configurations, the Ritz basis can be expressed in the following form:

\[
U = \begin{bmatrix} U_j \end{bmatrix} = \begin{bmatrix} I_j \ 0 \end{bmatrix} \begin{bmatrix} U_j \ c \end{bmatrix}
\]

Where : \( c \in \mathbb{R}^{P_i} \), \( \psi \in \mathbb{R}^{l_i} \), \( \phi \in \mathbb{R}^{P} \) are respectively the generalized coordinate vector and the static and dynamic sub-basis which are function of the type of the interfaces configurations [8].

The passage in physical coordinate consists to eliminate the generalized coordinate \( c \) in the basis (14):

\[
U = \begin{bmatrix} U_j \\ U^m_j \\ U^i_j \end{bmatrix} = \begin{bmatrix} I_j \\ 0 \\ I_{\omega} \end{bmatrix} \begin{bmatrix} U_j \\ U^m_j \\ U^i_j \end{bmatrix} = \left[ T_0 \right] \{ U^c \}
\]

We propose to extend the transformation \( T_0 \) in the following form :

\[
[T] = \left[ T_0 \ \Delta T \right]
\]

Where : \( \Delta T = E \left\{ R_\Delta (\theta) \right\} \) is the unknown variation of the basis due to the stochastic terms \( \Delta Z \) (\( E \) is the first moment).

With the aim of generating a force basis associated to stochastic modifications, we approach the random response of the modified system by the response of the initial system :

\[
f_\Delta \left( \omega, \theta \right) = \Delta Z \left( \omega, \theta \right) U \left( \omega, \theta \right) \approx \Delta Z \left( \omega, \theta \right) U_0 \left( \omega \right)
\]

For each stochastic zone \((i)\), we define a force sub-basis \( F_{\Delta i} \) from the initial modal properties \( (Y_\omega, A_\omega) \) and the stiffness and mass matrices of the stochastic zones \( K_i(\theta) \) and \( M_i(\theta) \):

\[
F_{\Delta i}(\theta) = \sum_{k=1}^{Q} K_i^j \xi_k Y_\theta \left| \sum_{k=1}^{Q} M_i^j \xi_k Y_\theta A_\theta \right|
\]

We can obtain the representative force basis of the stochastic modifications group by the concatenation of the sub-basis \( F_{\Delta i}(\theta) \). We obtain finally the random residual vectors basis :

\[
R_\Delta (\theta) = R F_\Delta (\theta) \quad \text{with} \quad R = K_\theta - Y_\theta A_\theta Y_\theta^T
\]

5. PROJECTION ON THE POLYNOMIAL CHAOS

We take an interest in the projected calculation of the frequency responses for the structures with a linear behavior and lowly damped, submitted to deterministic excitation \( f_e \). The expansion of the random response \( u(\theta) \) in the polynomial chaos (PC), can be written in the following truncated form :

\[
U(\theta) = \sum_{i=1}^{N} u_i \Psi_i
\]

The SFEM method consists to represent each component \( U^i(\theta) \) (random variable of the unknown law) by a polynomial development in a standard normal random variables :

\[
U(\theta) = \sum_{i=1}^{P} u_j \Psi_j \left\{ \xi_k(\theta) \right\}^W_{k=1}
\]
where \( \{ \xi_i(\theta), i = 1, \ldots, M \} \) are the standard normal random variables which are used to discretize the random field describing the data, \( \Psi_j^\prime \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right) \) are the multidimensional Hermite polynomials defined from a set of \( M \) random variables \( \xi_i \); \( \{ u_j \} \) and \( P \), are respectively the coefficients and the development order of the expansion \((P = (M + p)! / (M! \times p!))\).

The expansion of the condensed displacement \( U^c(\omega, \theta) \) in the PC can be written in the following form :

\[
U^c(\theta) = \sum_{n=0}^{p} u_n^c \Psi_n \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right)
\]  

(22)

Similarly, the vector \( f^c_\Delta(\omega, \theta) \) can be written as :

\[
f^c_\Delta(\omega, \theta) = \sum_{n=0}^{p} \Delta Z^c_n u_n^c \Psi_n \left( \left\{ \xi_k(\theta) \right\}_{k=1}^M \right)
\]  

(23)

The insertion of the relations (22-23) in the equation (13), leads to the following equation :

\[
\sum_{n=0}^{p} Z_0^c N\Psi_n + \sum_{n=0}^{p} \sum_{q=1}^{P} (-\omega^2 M^c_n + j \omega B^c_n + K^c_n) \xi_n \Psi_n = f^c(\omega)
\]

(24)

The projection of the equation (24) on the polynomials \( \Psi_m \) (\( m = 0, \ldots, N \)), conducts to the following linear system :

\[
\sum_{n=0}^{p} Z_0^c N\Psi_m + \sum_{n=0}^{p} \sum_{q=1}^{P} (-\omega^2 M^c_n + j \omega B^c_n + K^c_n) u_n^c \left( \xi_q \Psi_n \Psi_m \right) = f^c(\Psi_m)
\]  

(25)

The projection of the PC is considered as a model condensation; we obtain finally a double model condensation \((T \times T_{PC})\). We can calculate after the first moments (means and standard deviation).

In view to calculate the condensed eigenmodes of the model associated to equation (25), we exploit the technique [9] where the following condition of orthogonality is verified :

\[
\Phi^T(\theta) \left( M_0 + \sum_{q=1}^{Q} M^c_q \xi_q \right) \Phi(\theta) = \delta_{vo} \quad ; \quad \Phi^T(\theta) \left( K_0 + \sum_{q=1}^{Q} K^c_q \xi_q \right) \Phi(\theta) = \omega_0^2(\theta) \delta_{vo}
\]

(26)

\( \omega_0^2(\theta) \), \( \Phi(\theta) \) represents the \( \nu \)th random eigenmode which is expressed in the truncated basis of the PC by the following expressions :

\[
\omega_0^2(\theta) = \omega_0^2 \sum_{n=0}^{p} a_n^r \Psi_n \quad ; \quad \Phi(\theta) = \sum_{n=0}^{p} \lambda_n^p \Psi_n \Phi_{0p}
\]  

(27)

Where, we suppose that the decomposition of \( \Phi(\theta) \) on the deterministic modal basis is possible.

The substitution of (27) in (26) and the projection on the polynomials \( \Psi_m \), allows finally the obtaining of a nonlinear system of order \( P(P+1)(N+1) \) as function of \( (a_n, \lambda_n^p) \).

6. NUMERICAL SIMULATION

The proposed simulation example concerns a frame structure (Figure 1) which is discretized by a bidimensional beams elements (3 DOF per node \( U_x, U_y, \theta_z \)). The FEM contains 162 DOF. The mechanical and geometrical characteristics are given by : \( b = 5 \times 10^{-3} \text{ m} \); \( h = 10^{-2} \text{ m} \); Area = \( b \times h \); \( E_0 = 2,1 \times 10^{11} \text{ N/m}^2 \); \( \rho_0 = 7800 \text{ kg/m}^3 \); \( \nu = 0,3 \)
The dynamic analysis is realized in the frequency band [0 – 500 Hz] including the first nine global eigenmodes. The structure is submitted to a localized excitation force at the node \( N_f \) according to the DOF \( U_x \). The observation point is considered at node \( N_0 \) according to the direction \( U_x \).

The structure is considered with two uncertain parameters per zones (modulus of elasticity at the foot of the vertical beams and the thickness throughout the transom). Note that uncertainties are introduced per zones, the SFEM is applied only in these zones and the rest of the structure will be deterministic.

By using the direct condensation method CB, the initial model with 162 DOF is reduced to condensed one with 21 DOF. The enrichment of the Craig-Bampton transformation basis is realized by seven random statistic residual vectors. We compare in table 1, the first ten random eigenmodes and those of the reference model. Figures (2 to 5) illustrate the evolution of the mean and the standard deviation of the random response as a function of the excitation frequency for the dispersions : \( \delta_E = 5\% ; \delta_h = 0,5\% \).

Figures (2-3), show the results of the MC Simulation for the condensed model (enriched basis : CBE) which are compared to those of the reference MC Simulation without condensation. These results show that the use of CBE basis allows a good dynamic representation throughout the frequency band [0 – 500 Hz]. The quality of the reduced model CBE compared with the reference is equally highlighted by the differences graphs illustrated on the same figure.

Figures (4-5) shows that a good coincidence is obtained between the mean and the standard deviation plots of the random responses which we calculated by the PC and the MC Simulation for 1000 draws.

Note that, when the dispersion factor increase, the order of PC influences the reconstitution quality of the response.

In order to highlight the interest of the reduced models, we compare the CPU time (table 2) between the condensed and reference models. The results justify the interest of the proposed double condensation (CBE + PC).

7. CONCLUDING REMARKS

In this article, we propose a new strategy which consists to couple the SFEM and a robust condensation method in view of optimizing the behaviour of dynamics structures presenting local uncertainties per zones or per substructures. The Ritz basis is enriched by additional vectors which are chosen reasonably. These vectors are obtained from the random static loadings representative of the modifications forces. This enriched basis allows the construction of a reduced model which is robust towards uncertain structural modifications.

The comparison of the calculation costs allows to highlight the interest of the double condensation method followed by the projection on polynomial chaos.

This method is able to respond, in terms of reducing the size of the FEM and predicting the needs of the reanalysis encountered in the iterative optimization procedures of the dynamic behaviour of stochastic finite elements models with large size.

The current works concerns : the study of the effect of the parametric uncertainties levels on the order of the polynomials chaos ; the adaptation of the proposed method to the dynamic substructuring and the statistic link between the proposed method and the non parametric approaches.

REFERENCES


Figure 1: FEM of a planar frame structure

<table>
<thead>
<tr>
<th>Mean eigenfrequency (Hz)</th>
<th>Random eigenfrequency (Hz) (first moment)</th>
<th>CB</th>
<th>CBE</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>$\varepsilon_f$ %</td>
<td>$\varepsilon_U$ %</td>
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<tr>
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</tr>
</tbody>
</table>

Table 1: Precision of the eigenfrequencies ($\varepsilon_f$) and the eigenvectors ($\varepsilon_U$)
Figure 2: Means of the random response at the position « NO ». MC Simulation with 1000 samples, reference and condensed models

Figure 3: Standard deviation of the random response at the position « NO ». MC Simulation with 1000 samples, reference and condensed models

Figure 4: Means of the random response at the position « NO ». PC order 4 and MC simulation with 1000 samples, condensed model

Figure 5: Standard deviation of the random response at the position « NO ». PC order 4 and MC simulation with 1000 samples, condensed model

<table>
<thead>
<tr>
<th>Dispersion of uncertain parameters</th>
<th>CPU (min)</th>
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<tr>
<td>h 0.5%</td>
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Table 2: CPU for reference model and condensed model