Modal Interaction of Random Dynamical Systems

Roger Ghanem
Professor
Department of Civil Engineering
The Johns Hopkins University
3400 N. Charles Street
Baltimore, MD 21218, USA
Email: ghanem@jhu.edu

Debraj Ghosh
Graduate Student
Department of Civil Engineering
The Johns Hopkins University
3400 N. Charles Street
Baltimore, MD 21218, USA
Email: debrajg@jhu.edu

ABSTRACT

An analysis of the statistical behavior of the modal parameters of linear dynamical systems with uncertain physical properties is presented here that focusses on quantifying the modal interactions in such systems. In the current literature the covariance matrices of the modal vectors are usually reported as informative regarding modal interactions. Here a new method of representing these modal interaction is proposed using the polynomial chaos expansion according to which a stochastic process is represented with respect to a hilbertian basis consisting of the multidimensional Hermite polynomials in orthogonal gaussian variables. The statistical vibrational modes of the stochastic system are projected on the space spanned by the modal vectors of the deterministic system and the statistics of the projections are computed. In this manner, we get the contributions of each mode of the deterministic system to the modes of the system with random properties. Finally, statistics of these contributions are computed. In the presence of uncertainty, this representation will be helpful in selecting the width of frequency band for designing a system, predicting the response configuration and identifying some potentially dangerous loading configurations. A numerical example is given to elaborate the method.

Notation

\( \tilde{\phi}_i \) \( i^{th} \) mode of the deterministic system
\( \phi^{(l)} \) \( l^{th} \) mode of the variable system
\( \phi^{(l)}_k \) \( k^{th} \) chaos component of \( l^{th} \) mode
\( e_i \) Random coefficient
\( \xi_i \) Standard normal random variable
\( K_x, K_y, K_z \) Spring stiffness
\( \lambda_i \) Eigenvalue
\( \phi_i \) Eigenvector
\( P \) Maximum index of polynomial chaos
\( \psi_i \) \( i^{th} \) polynomial chaos

\(^1\text{Corresponding author}\)
Introduction

Modal analysis is the most common approach to characterizing the behavior of dynamical systems. It is equally significant in the corresponding computational and experimental analyses. The impact of variability, stemming from fabrication and subscale features, on the predicted modal content of these systems is clearly of great interest for their safe and economical design. Traditional approaches to analyzing the effect of variability on the mode shapes and the natural frequencies have taken the form of sensitivity analysis, permitting the identification of the relative significance of various sources of scatter on design quantities. In the present paper, a probabilistic approach is adopted that relies on the probabilistic representation of the variability and the associated dynamics. In particular representations of the mode shapes and the natural frequencies of the systems are obtained that permit a novel perspective on statistical modal analysis, introducing the concept of statistical modal interaction. The proposed methodology and interpretation have significant impact on the development of reduced-order models of dynamical systems that permit their safe design at specified levels of confidence.

Polynomial Chaos Representation

The Polynomial Chaos decomposition is a representation of a random variable or stochastic process with respect to a Hilbertian basis consisting of the multidimensional Hermite polynomials in orthogonal Gaussian variables [1]. Any eigenvalue and eigenvector of the variable system with \( n \)-degrees of freedom can be expanded according to this basis [3]

\[
\phi = \sum_{i=0}^{P} \psi_i \phi_i, \quad \lambda = \sum_{i=0}^{P} \psi_i \lambda_i
\]

where \( \xi_i \) are standard normal random variables and \( \psi_i \) are zero-mean, orthogonal polynomials in \( \xi_i \) such that,

\[
\psi_0 \equiv 1, \quad \langle \psi_i \rangle = 0, \quad \langle \psi_i \psi_j \rangle = 0, \quad i \neq j
\]

The current technique used for calculating the chaos coefficients \( \phi_i \) and \( \lambda_i \) is a non-intrusive (simulation based) method. In this method \( \phi_i \) and \( \lambda_i \) are calculated using their generalized Fourier coefficient expressions,

\[
\phi_i = \frac{\langle \phi \psi_i \rangle}{\langle \psi_i^2 \rangle}, \quad \lambda_i = \frac{\langle \lambda \psi_i \rangle}{\langle \psi_i^2 \rangle}.
\]

The denominator of the above expressions can be readily obtained analytically or numerically [1], while the numerator can be evaluated by Monte Carlo simulation. For each
realization of the set of random variables $\xi_i$, the corresponding eigenproblem is solved for $\lambda$ and $\phi$, and the realization of $\psi_i$ is simultaneously computed. Over a large number of realizations, $\phi_i$ and $\lambda_i$ are calculated by taking the arithmetic average of the inner product of the terms in the numerator and dividing by the denominator. Statistics of the eigensolutions are calculated by,

$$\bar{\lambda} \equiv \langle \lambda \rangle = \lambda_0, \quad \sigma_{\lambda} = \sqrt{\frac{\sum_{i=1}^{P} \langle \psi_i^2 \rangle \lambda_i^2}{\sum_{i=1}^{P} \langle \psi_i^2 \rangle}}$$

and

$$\bar{\phi} \equiv \langle \phi \rangle = \phi_0$$

A characterization of the higher order statistics of modal vectors is discussed in the next section.

**Modal Interaction**

While reporting the statistics of the modal vectors, usually their mean and standard deviation are presented. Here a novel representation of the modal statistics is described. In this methodology, the dynamic modes of the uncertain system are expressed in terms of the dynamic modes of the deterministic or mean system. This representation has a number of advantages. In particular, the concept of statistical modal overlapping, or statistical leakage, can be readily introduced. Since for most systems, the eigenmodes of the deterministic system are used as the benchmark for design and decision making, describing the behavior of the stochastic system in terms of the modes of the deterministic system, provides an easy interpretation of their significance. This becomes more useful if the same system is analyzed for different types and levels of uncertainty.

Thus, expressing the $l$th statistical physical mode of the stochastic system as a linear combination of the deterministic physical modes, results in,

$$\phi^{(l)} = \sum_{i=N_1}^{N_2} e_{ij} \bar{\phi}_i,$$

and the $l$th mode of the variable system can be represented in its polynomial chaos expansion as

$$\phi^{(l)} = \sum_{k=0}^{P} \psi_k \phi_k^{(l)}$$

The $k$th chaos component of the $l$th mode can be expressed as a linear combination of the deterministic physical modes as

$$\phi_k^{(l)} = \sum_{i=N_1}^{N_2} C_{kl} \bar{\phi}_i$$
where $C_{kl}^{i}$ are some coefficients to be determined by solving the set of $n$ linear equations. Using the equations (2) and (3) we get,

$$
\phi^{(l)} = \sum_{k=0}^{P} \sum_{i=N_1}^{N_2} \psi_k C_{i}^{kl} \bar{\phi}_i
$$

$$
= \sum_{i=N_1}^{N_2} (\sum_{k=0}^{P} \psi_k C_{i}^{kl}) \bar{\phi}_i
$$

(4)

Comparing equations (1) and (4) results in,

$$
e_{l}^{i} = \sum_{k=0}^{P} \psi_k C_{i}^{kl} .
$$

(5)

The statistics of $e_{l}^{i}$ provide a statistical measure of the contribution of the deterministic modes towards the modal properties of the variable system.

$$
\langle e_{l}^{i} \rangle = \sum_{k=0}^{P} \langle \psi_k \rangle C_{i}^{kl} = C_{i}^{0l}
$$

(6)

$$
\langle (e_{l}^{i})^2 \rangle = \sum_{k=0}^{P} \langle \psi_k^2 \rangle (C_{i}^{kl})^2
$$

(7)

Here it can be noted that $\langle e_{l}^{i} \rangle$ is the projection of the 0th chaos component of $l$th mode (i.e. the $l$th mean mode of the variable system found by chaos decomposition) on the $i$th mode of the deterministic system. Clearly, this should tend to Kronecker delta, $\delta_{il}$ as the level of system variability decreases.

**Numerical Example**

To explain the above mentioned procedure, a numerical example is provided here. The model consists of two cantilever beams inter-connected at their free end by a spring (Fig 1) whose two transverse stiffnesses have a value equal to 0.1% times that of the axial stiffness. The finite element modelling is done using Salinas [4]. Eight-noded HEX8u elements are used in the computational model. The modal interaction is analyzed in two different regimes, the first one involving the first five modes while the second one is between 16th to 20th modes. The selection of these regimes is arbitrary here and is meant to demonstrate the role of modal interactions in different part of the frequency spectrum. It will be observed that the level of statistical leakage depends on the frequency bandwidth in which the system is operating. The density of the beam material and the spring stiffness are considered as uncertain. They are modelled as follows

$$
\gamma = \bar{\gamma} + \sigma_1 \xi_1
$$

$$
K_x = \bar{K}_x + \sigma_2 \xi_2 , K_y = \bar{K}_y + \sigma_2 \xi_2 , K_z = \bar{K}_z + \sigma_2 \xi_2
$$
where $\bar{\gamma}$ and $\bar{K}$ are the mean values of density and stiffness, $\sigma_1$ and $\sigma_2$ are the standard deviations, $\xi_1$ and $\xi_2$ are two independent standard normal variables.

The first 5 eigenvalues of the corresponding deterministic system are 0.29, 0.52, 4.10, 5.07, 9.54 and the 16th to 20th eigenvalues are 242.14, 295.75, 372.42, 486.6, 514.74. For calculating the statistics of $e_l^i$ in the first regime, (1st to 5th modes), $l$ is considered from 1 to 5, $N_1 = 1$, $N_2 = 5$, and for the second regime (16th to 20th modes), $l$ is considered from 16 to 20, $N_1 = 16$, $N_2 = 20$ in equations 6 and 7. Here a standard deviation of 20% of the mean is assumed for both beam material density and spring stiffnesses. The chaos coefficients are computed using 2000 realizations. Up to 2nd order chaos expansion is used to compute the statistical leakage between the eigenvectors. The results are tabulated below.

\[
\begin{array}{cccccc}
1.0165 & -0.0000 & -0.0002 & 0.0000 & -0.0001 \\
0.0000 & 1.0167 & -0.0000 & -0.0005 & 0.0000 \\
0.0001 & -0.0000 & 1.0161 & 0.0000 & 0.0000 \\
-0.0000 & 0.0006 & -0.0000 & 1.0165 & -0.0000 \\
-0.0001 & -0.0000 & -0.0004 & 0.0000 & 1.0166 \\
\end{array}
\]

Table 1: $\langle e_l^i \rangle$, 1st to 5th mode

In the tables above, the columns refer to the statistical dynamical modes and the
rows correspond to the modes of the deterministic system. From table 1 and 2, it is observed that the matrix in this table is significantly dominated by its diagonal terms. This suggests that in this case, the modal interaction is very low. More specifically, in this regime of operation, mode shapes of the uncertain system will look like those of the deterministic system. Table 3, associated with a higher-frequency regime, shows significant leakage between the statistical and deterministic modes. The third column, for instance, represents the 18th mode of the uncertain system. It is observed that the second and fifth rows (those corresponding to the 16th and 20th modes of the deterministic system respectively) contribute to the magnitude of the 18th mode. This suggests that the 18th mode of the uncertain system may have some contribution from the 16th and 20th mode of the deterministic system. In other words, for a particular realization of the uncertain system, the 18th mode will look like a “mixed mode” with contributions from 16th, 18th and 20th mode of the average deterministic system.

It is observed that the modal interactions are not uniform throughout the frequency spectrum. A particular system may or may not exhibit significant modal interactions depending upon the spectral band of interest. Also it can be shown that the magnitude of the modal interaction depends on the level of uncertainty. Here the truncated set of modes are considered for representing the interaction, i.e. the contribution from $N_2 - N_1 + 1$ deterministic modes are taken into account. This reduction or local representation is justified under the assumption that the mode shapes of a perturbed system are not significantly different from those of the unperturbed system. But this may not be the case for some structures like disordered periodic structure, where spatial localization of mode shapes is observed. In that case, a wider bandwidth may be required for describing the statistical modes i.e. more number of deterministic modes should be taken. The same is applicable for any structure with high variability.
Conclusion

A tool for characterizing and representing the modal interactions of uncertain systems is proposed. This representation gives the contribution (in statistical sense) of the modes of the deterministic system to the modes of the uncertain system. The method uses the polynomial chaos expansion which is an effective tool for characterizing the modal parameters, even for high levels of uncertainty [2]. This representation of modal interaction is different from the traditional one obtained from the mean and standard deviation of the modal vectors in that it gives an idea of how the modal mixing in the presence of uncertainty when the deterministic model is available as the baseline. Thus, given these modal interaction tables, the behavior of the uncertain system can be represented in terms of that of the deterministic model.

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References


