On a Theory of the Quantification of the Lack of Knowledge (LOK) in Structural Computation

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ABSTRACT

Still today, some of the phenomena which occur in structural dynamics cannot be described, even using a properly updated model: some uncertainties remain. This paper presents the basic concepts of a new theory which uses the concept of Lack of Knowledge (LOK) to model the uncertainty using two bounds of the energy in each substructure. The inverse problem, which consists in determining the Lack of Knowledge for each substructure from test results on the whole structure, has been studied on a simple example.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>A real structure from a family of similar ones</td>
</tr>
<tr>
<td>$E$</td>
<td>A substructure of $\Omega$</td>
</tr>
<tr>
<td>$K_E$</td>
<td>Stiffness matrix of $E$</td>
</tr>
<tr>
<td>${\phi_i, \omega_i}$</td>
<td>Eigenmode/eigenfrequency (rd/s) pair</td>
</tr>
<tr>
<td>$\epsilon_E$</td>
<td>Strain energy of $E$</td>
</tr>
<tr>
<td>$(m_E^+, m_E^-)$</td>
<td>Basic Lack of Knowledge for $E$</td>
</tr>
<tr>
<td>$(\omega_i^+, \omega_i^-)$</td>
<td>Bounds of the basic Lack of Knowledge for $E$</td>
</tr>
<tr>
<td>$F(m_E^+, m_E^-)$</td>
<td>Cost function for the identification process</td>
</tr>
<tr>
<td>$\omega_i^{\text{low}}$</td>
<td>99%-lower bound of the experimental $\omega_i$</td>
</tr>
<tr>
<td>$\omega_i^{\text{upp}}$</td>
<td>99%-upper bound of the experimental $\omega_i$</td>
</tr>
<tr>
<td>$\omega_i^{\text{low}} (m_E^+, m_E^-)$</td>
<td>99%-lower bound of the LOK-modelled $\omega_i$</td>
</tr>
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<td>$\omega_i^{\text{upp}} (m_E^+, m_E^-)$</td>
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</table>

1 INTRODUCTION

Today, the problem of quantifying the quality of a mechanical model remains a major issue. As far as the comparison with an experimental reference is concerned, many approaches can be used to update a dynamic model (stiffness, mass and damping) based on free or forced vibration tests\cite{1,2}. However, some phenomena cannot be described even by a properly updated model: as can be seen on Figure 1, the distinct values of the eigenfrequencies calculated using a theoretical model are hardly representative of the distribution of eigenfrequencies measured experimentally.

In this paper, these uncertainties – which, overall, cause a general deterministic model to be inefficient – are treated as Lacks of Knowledge (LOKs), a concept initially introduced at the EADS Launch Vehicles Division, which is developed below.

To model these uncertainties, two general options can be followed: a first approach consists in defining a lack of knowledge by an interval: this deterministic approach, introduced by Ben-Haim\cite{3,4}, defines info-gap models, which quantify the uncertainty as the "size of
the disparity between what is known and what could be known and make the use of probabilistic laws unnecessary, which is desirable in situations where information is severely lacking; however, the simple addition of the uncertainties can lead to an overestimated result and be very unrealistic.

As an alternative, probabilistic methods have been increasingly introduced. Generally, these methods consist in studying the effects of the uncertainties affecting the input on the variability of the output, but this can be done in various ways. For example, Gaul et al.[5-6] use fuzzy parameters to quantify the overall uncertainty of the model’s output, whereas Hemez et al.[7-8] build meta-models by spanning the space of the most influential parameters and using a specific technique to reduce the computational effort drastically.

The theory presented here was introduced in[9]. In a sense, it follows both of the above approaches. It is based on the concept of Lack of Knowledge (LOK) applied to the strain energy distribution in the structure; at this point, we assume there is no error in the distribution of the mass. The basic characteristics of the Lack of Knowledge on a substructure consist of two scalar internal variables which, in a global sense, bound the strain energy of the substructure; these two variables, called basic LOKs, follow probabilistic laws. Having identified the basic characteristics, one can calculate the Lack of Knowledge for any quantity associated with the solution. In this paper, the concept of Lack of Knowledge is developed in the field of vibration problems. At this stage, the joints are also treated as substructures.

2 BASIC LOKS

For each substructure $E$, two positive numbers $m^E_+$ and $m^E_-$ are defined and constitute the two bounds of the energy distribution in the substructure:

$$\left(1 - m_\pm^E\right) \pi_E \leq e_E \leq \left(1 + m_\pm^E\right) \pi_E$$

(1)

where, for Substructure $E$, $e_E$ and $\pi_E$ are the strain energies linked to the real structure and to the theoretical model respectively. $m^E_+$ and $m^E_-$ are the upper basic LOK and the lower basic LOK respectively.

The basic LOKs are sampled using a probabilistic model; the nature of this model is chosen a priori and its characteristics are given by two bounds $m^E_+$ and $m^E_-$, respectively called the upper bound and the lower bound of the basic LOK:

- if a uniform distribution is chosen, these two bounds include all the values of $m^E_+$ and $m^E_- -
- if a normal distribution is chosen, one can decide that these two bounds include 99% of the possible values of $m^E_+$ and $m^E_-$. In the absence of specific information, it is reasonable to choose a uniform distribution. A normal distribution is appropriate for cases in which the sources of errors are material uncertainties.

In summary, one can say that the basic LOKs $(m^E_+, m^E_-)$ follow a stochastic distribution such that:

$$-m^E_+ \leq -m^E_- \leq 0 \leq m^E_+ \leq m^E_-$$

(2)

3 THE CONCEPT OF LOK

The concept of LOK is based on the discrepancies between the results obtained from the model and the quantities measured experimentally. The experimental data come from a family $\Omega$ of actual structures which are similar, but not necessarily identical.

Let us consider a certain quantity of interest, $\alpha$.

- based on the real structures, we can derive two 99%-bounds $\alpha^\text{low}_\exp$ and $\alpha^\text{upp}_\exp$, which include 99% of the experimental values $\alpha^\text{exp}$.
- based on samples of $(m^E_+, m^E_-)$, one can use the model to calculate two bounds $\alpha^* - \alpha^*$ of $\alpha$ and their probabilistic distribution. (This will be shown in Section 4.) Then, one can derive two 99%-bounds by taking many samples of $(m^E_+, m^E_-)$: these bounds, denoted $\alpha^\text{low}$ and $\alpha^\text{upp}$, include 99% of the values of $\alpha^* - \alpha^*$.

The basic LOKs must be such that for any quantity of interest $\alpha$

$$\alpha^\text{low} \leq \alpha^\text{low}_\exp \leq \alpha^\text{upp}_\exp \leq \alpha^\text{upp}$$

(3)

Moreover, the intervals defined by the basic LOKs must be as small as possible: if one of the LOKs is reduced, one of the relationships (3) is no longer true.

4 EFFECTIVE LOKS

Equations (1) and (2) are the basic relationships which make the identification of the basic LOKs from the experimental data possible. Of course, the comparison between the model’s results and reality is made not directly through these equations, but using quantities which are standard in the modal analysis domain: in this paper, we use free-vibration tests; therefore, our quantities of interest $\alpha$ are eigenfrequencies and eigenmodes. The previously defined quantities $\alpha^\text{low}$ and $\alpha^\text{upp}$ are called effective LOKs and their values for eigenfrequencies and eigenmodes are given in the following sections.

4.1 Effective LOK for an eigenfrequency

If the modes are mass-normalized, the eigenfrequency $\omega_i$ verifies:

$$\Delta \left( \omega^\text{low}_i \right) = \omega^\text{low}_i - \omega^\text{upp}_i = \phi_i^T K \phi_i - \phi_i^T \bar{\Sigma} \phi_i$$

(4)

which, assuming that $\phi_i \simeq \bar{\phi}_i$, is approximated by:

$$\Delta \left( \omega^\text{low}_i \right) = \bar{\phi}_i^T \left( K - \bar{\Sigma} \right) \bar{\phi}_i = 2 \sum_{E \in \Omega} \left( e_E \left( \bar{\phi}_i \right) - \pi_E \left( \bar{\phi}_i \right) \right)$$

(5)
With the fundamental equation (1), we get:

\[-2 \sum_{k \in \Omega} m_k \tau_k(\bar{\phi}_i) \leq \Delta (\omega_i^2) \leq 2 \sum_{k \in \Omega} m_k \tau_k(\bar{\phi}_i)\]  

(6)

Thus, using the theoretical model and the LOK concept, we can derive two bounds for the quantity \( \Delta (\omega_i^2) \); these bounds depend on the values of \( m_k^+ \) and \( m_k^- \) sampled using the selected probabilistic model and, therefore, on the values of \( m_k^+ \) and \( m_k^- \). Typically, we consider that these bounds should include 99% of the calculated values of the eigenfrequencies.

### 4.2 Effective LOK for an eigenmode

For small values of the basic LOKs, we can approximate the variation of an eigendisplacement by:

\[ \phi_{i} - \bar{\phi}_i = U^T \Delta \bar{\phi}_i = \sum_{E \in \Omega} U^T (K_E - K_{\bar{E}}) \bar{\phi}_i \]  

(7)

where \( U \) is a given vector.

Using this relationship, we can derive the effective LOK for an eigenmode by writing:

\[ U^T K_E \bar{\phi}_i = \frac{1}{2} e_E(U + \bar{\phi}_i) - \frac{1}{2} e_E(U - \bar{\phi}_i) \]  

(8)

With the fundamental equation (1), this leads to:

\[ \phi_{i} - \bar{\phi}_i \leq \frac{1}{2} \sum_{E \in \Omega} \{ m_k^+ \tau_k(U + \bar{\phi}_i) + m_k^- \tau_k(U - \bar{\phi}_i) \} \]  

(9a)

\[ \phi_{i} - \bar{\phi}_i \geq -\frac{1}{2} \sum_{E \in \Omega} \{ m_k^+ \tau_k(U + \bar{\phi}_i) + m_k^- \tau_k(U - \bar{\phi}_i) \} \]  

(9b)

Exactly as we did for the eigenfrequencies, we can derive from the previous relationships the effective LOK defined by the two bounds which include 99% of the sampled values.

### 5 IDENTIFICATION OF THE BASIC LOKS

#### 5.1 Principle

The purpose of identifying the basic LOKs is to find the values of \( m_k^+ \) and \( m_k^- \) which are the most representative of the dispersion of the experimental data. This dispersion can be described by 99%-bounds for any quantity of interest:

- \( \omega_i^{2\text{low}}(m_k^+, m_k^-) \) and \( \omega_i^{2\text{upp}}(m_k^+, m_k^-) \) for the square of the \( i^{th} \) eigenfrequency;
- \( \phi_i^{\text{low}}(m_k^+, m_k^-) \) and \( \phi_i^{\text{upp}}(m_k^+, m_k^-) \) for the value at the \( k^{th} \) DOF of the \( i^{th} \) eigenmode.

Once the choice of the probabilistic laws has been made for each substructure, the basic LOKs are sampled in order to obtain the effective LOKs defined in Section 4. Let us recall that these numbers are the bounds which include 99% of the sampled values. Thus, for the model with LOKs, we have:

- \( \omega_i^{2\text{low}}(m_k^+, m_k^-) \) and \( \omega_i^{2\text{upp}}(m_k^+, m_k^-) \) for the square of the \( i^{th} \) eigenfrequency;
- \( \phi_i^{\text{low}}(m_k^+, m_k^-) \) and \( \phi_i^{\text{upp}}(m_k^+, m_k^-) \) for the value at the \( k^{th} \) DOF of the \( i^{th} \) eigenmode.

The identification process consists in finding the values of \( m_k^+ \) and \( m_k^- \) defining the smallest possible intervals such that:

\[ \omega_i^{2\text{low}}(m_k^+, m_k^-) \leq \omega_i^{2\text{low}}(m_k^+, m_k^-) \leq \omega_i^{2\text{upp}}(m_k^+, m_k^-) \]  

(10a)

\[ \phi_i^{\text{low}}(m_k^+, m_k^-) \leq \phi_i^{\text{low}}(m_k^+, m_k^-) \leq \phi_i^{\text{upp}}(m_k^+, m_k^-) \]  

(10b)

#### 5.2 Solution of the inverse problem

In order to minimize this interval, let us introduce the following cost function:

\[ J(m_k^+, m_k^-) = \frac{1}{4p} \sum_{i=1}^{p} \left\{ \frac{\omega_i^{2\text{low}}(m_k^+, m_k^-) - \omega_i^{2\text{low}}(m_k^+, m_k^-)}{\omega_i^{2\text{low}}(m_k^+, m_k^-)} \right\} + \frac{1}{4p} \sum_{i=1}^{p} \left\{ \frac{\omega_i^{2\text{upp}}(m_k^+, m_k^-) - \omega_i^{2\text{upp}}(m_k^+, m_k^-)}{\omega_i^{2\text{upp}}(m_k^+, m_k^-)} \right\} \]

(11)

where \( p \) is the number of mode shapes being taken into account. Thus, the identification process consists in minimizing \( J(m_k^+, m_k^-) \) under the constraints (10a) and (10b).

Of course, this identification process may not be straightforward: a particular cost function could be added in order to regularize the inverse problem: for example, one can choose the following cost function, which tends to make the level of Lack of Knowledge uniform throughout the whole structure:

\[ G(m_k^+, m_k^-) = \sup_{E \in \Omega} \left| m_k^+ - m_k^- \right| + \sup_{E \in \Omega} \left| m_k^- - m_k^+ \right| \]  

(12)

### 6 APPLICATION TO A SIMPLE PROBLEM

#### 6.1 Definition of the structure

The structure being considered is the plane truss of Figure 2, which is made of four aluminium bars connected by spherical joints. A family of such trusses was simulated and their eigenfrequencies and eigenmodes formed the data which was used in the identification process described above.

In the associated model, we assumed that the bars were solicited only in traction-compression; moreover, the joints at Nodes 1 and
6.2 Identification of the basic LOKs for the structure considered

The identification of \((\vec{m}_{b23}^+, \vec{m}_{b23}^-, \vec{m}_{b23}^{++}, \vec{m}_{b23}^{--})\) was achieved using the first five modes and assuming a normal distribution for the LOKs \(m_{b23}^+\) and \(m_{b23}^-\) of Bar 2-3 and a uniform distribution for the LOKs \(m_{j1}^+\) and \(m_{j1}^-\) of the elastic joint at Node 1. The other substructures were assumed to be perfectly known. We obtained the minimum of (11) by spanning the space of the basic LOKs:

\[
\begin{align*}
\vec{m}_{b23}^+ &= 0.094 & \quad & \vec{m}_{b23}^- = 0.088 \\
\vec{m}_{j1}^+ &= 2.01 & \quad & \vec{m}_{j1}^- = 0.099
\end{align*}
\]

These values are in pretty good agreement with the probabilistic model used for the simulation of experimental data, i.e. a centered distribution with a 10% range for the stiffness of Bar 2-3 and a 5%-shifted distribution with a 15% range for the stiffness of the joint at Node 1.

6.3 Calculation of the effective LOKs

Then, with these identified values of \((\vec{m}_{b23}^+, \vec{m}_{b23}^-, \vec{m}_{b23}^{++}, \vec{m}_{b23}^{--})\), we calculated the effective LOKs for the next three modes (i.e. 6, 7 and 8) in order to test the results of the identification process. The basic LOKs were sampled with the identified values and the probabilistic laws chosen; the corresponding calculated 99%-bounds are listed and compared with the experimental 99%-bounds in Tables 1 and 2 below.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\omega_i^{\text{low}}) ((\text{rad/s}))</th>
<th>(\omega_i^{\text{exp}}) ((\text{rad/s}))</th>
<th>(\omega_i^{\text{upp}}) ((\text{rad/s}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4.73045 \times 10^9</td>
<td>4.73039 \times 10^9</td>
<td>4.73241 \times 10^9</td>
</tr>
<tr>
<td>7</td>
<td>4.81957 \times 10^9</td>
<td>4.81958 \times 10^9</td>
<td>4.82413 \times 10^9</td>
</tr>
<tr>
<td>8</td>
<td>1.12753 \times 10^{10}</td>
<td>1.12753 \times 10^{10}</td>
<td>1.12756 \times 10^{10}</td>
</tr>
</tbody>
</table>

TABLE 1: Comparison of the squared values of the eigenfrequencies for Modes 6, 7 and 8

<table>
<thead>
<tr>
<th>(i)</th>
<th>(k_{i,\text{low}})</th>
<th>(k_{i,\text{exp}})</th>
<th>(k_{i,\text{upp}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>−0.20852</td>
<td>−0.20845</td>
<td>−0.20840</td>
</tr>
<tr>
<td>7</td>
<td>−0.21054</td>
<td>−0.21034</td>
<td>−0.21023</td>
</tr>
<tr>
<td>8</td>
<td>−0.16512</td>
<td>−0.16509</td>
<td>−0.16504</td>
</tr>
</tbody>
</table>

TABLE 2: Comparison of the eigendisplacements for Modes 6, 7 and 8

The constraints (10a) and (10b) were successfully respected for Modes 6, 7 and 8, which shows the consistency of the identification obtained with the first five modes.

7 CONCLUSION

In this paper, we presented a new approach to the quantification of Lacks of Knowledge. The basic LOKs are defined as bounds of the energy distribution in substructures and the resulting effective LOKs enable the identification of the basic LOKs using quantities which are commonly accessible in modal analysis. This identification process was carried out successfully on a simple problem; more complicated cases and real structures will be considered in the near future.

The objective of this theory is to develop a general method for reducing the Lacks of Knowledge on a predetermined family of parameters by designating which tests should be performed or which models of substructures should be improved.

REFERENCES


