ABSTRACT

Dynamic interaction between two bodies, where one is significantly more flexible than the other, is common (e.g. rotor-casing contact). This paper presents a linear model for simulating contact between rigid and flexible bodies. The motion between contacts is predicted by modal superposition, using separate models for the two components. During contact, the models are coupled via a linear contact stiffness element. The coupled equations are solved by numerical integration. The presented results relate to the short-term dynamics during contact and focus on the effects of using different numbers of degrees of freedom in the model of the flexible body.

NOMENCLATURE

\[ [M], [C], [K] \]  Mass, damping, stiffness matrices
\{q\}  Vector of generalised coordinates
\{Q\}  Vector of generalised forces
\{\phi\}  Eigenvector matrix
\{\xi, \zeta\}  Modal coordinates of ring and pendulum
\{u\}  State vector
c  Rotor-ring radial clearance
F  Rotor-ring contact force
KC  Contact stiffness
V  Pendulum displacement at point of contact
WR, WT  Radial and tangential displacement of ring

1. INTRODUCTION

There are many situations involving dynamic interaction between two bodies, one flexible, and the other essentially rigid. An important example is the interaction between a rotor and a flexible casing in an aero-engine, where the effect of component flexibility is becoming increasingly important in the context of the drive for weight reduction. There is therefore a need to study the effect of component flexibility on contact dynamics.

In this paper, a linear model is used to simulate contact between a rigid body and a flexible component. The flexible body is an elastic ring and the rigid body is a spring-restrained pendulum. This simple model can be extended to simulate rotor-casing interaction. A linear spring (contact stiffness) is used to represent local deformation of the bodies while they are in contact. The motion between contacts is predicted efficiently by modal superposition, using separate ring and pendulum models. During contact the two models are coupled by the contact stiffness and the resulting equations are solved by numerical integration. This paper deals with the short-term response of the system during contact and considers the complexity of model that is required to give converged predictions.

2. THE SYSTEM

Figure 1 shows the system to be analysed, consisting of two sub-systems. The rigid pendulum is restrained by springs at the lower bearing, is pivoted at O and can move with one degree-of-freedom, \( \phi_Y \), about axis OY (Figure 2). The rotor disc on the end of the pendulum is located in a clearance space within a flexible ring, symmetrically supported by eight springs. The parameter values for the system considered in this study are shown in Table 1.

The displacement of the ring will be described using \( N \) eigen-functions of a free circular ring as the generalised coordinates. The radial and tangential displacements, \( w_R \) and \( w_T \) respectively, of a point on the ring at angle \( \theta \) from the OX direction are therefore

\[
\begin{align*}
    w_R(\theta) &= \sum_{n=1}^{N} \left[ q_{nc} \cos(n\theta) + q_{ns} \sin(n\theta) \right] \\
    w_T(\theta) &= \sum_{n=1}^{N} \frac{1}{\alpha_n} \left[ q_{nc} \sin(n\theta) - q_{ns} \cos(n\theta) \right]
\end{align*}
\]

\( q_{nc} \) and \( q_{ns} \) are the generalised coordinates associated with ring displacements which are, respectively, symmetric and anti-symmetric with respect to \( \theta = 0 \) and \( \alpha_n (=n) \) is the ratio of the maximum radial and tangential displacements for each value of \( n \). Note that \( n = 1 \) represents rigid body displacement of the ring and \( n > 1 \) represents elastic deformation of the ring (the symmetric deflection for \( n = 2 \)).
as illustrated in Figure 2. The question as to what value of \( N \) should be used when considering interaction between the pendulum and the ring is considered in detail later.

The individual equations of motion for the pendulum and the ring were derived using Lagrange’s equations. Reference (1) presents the mass and stiffness matrices in detail. These equations can be used to predict the non-contact motion of the individual components in terms of a modal summation. For the results presented in this paper, the pendulum is restricted to movement in the XZ-plane. Contact between the pendulum and ring therefore only occurs in the XZ-plane, i.e. at \( \theta = 0^\circ \) or \( \theta = 180^\circ \). Consequently, due to the assumed symmetry of the ring support springs, only those generalised coordinates associated with symmetrical displacements of the ring (i.e. \( q_{nc} \)) will have non-zero values.

Clearly, contact between the pendulum and ring will occur when the algebraic difference between the pendulum displacement and the radial displacement of the ring at \( \theta = 0^\circ \) or \( \theta = 180^\circ \) exceeds the initial clearance. In general, the displacement of the ring will consist of a combination of rigid body displacement and elastic deformation. It will be assumed here that the elastic deformation of the ring is small relative to the initial clearance so that the point of contact will be determined with good accuracy by considering the rigid body displacements only. If however the elastic deformation of the ring is relatively large compared to the initial clearance then a different approach to finding the contact point is required. This is discussed in more detail in reference (1).

3. CONTACT BEHAVIOUR

3.1 Contact Stiffness

The force-displacement relationship between elastic bodies in contact is inherently non-linear. Nevertheless, it is beneficial to linearize the relationship in the interests of computational efficiency. For the system considered, the Hertzian force-displacement relationship for the circular contact between a sphere and a flat plate, reference (2), was used as the basis for calculating a linear contact stiffness to model the local contact deformation. Figure 3 shows the non-linear force displacement relationship for circular contact, for a relevant range of contact forces. The linear approximation to the circular contact curve (stiffness \( K_C = 250 \text{ MN/m} \)) used in the present study is also shown.
3.2 Predicting Motion During Contact

The individual mathematical models of the ring and pendulum, fully describe the non-contact dynamics of each component. When contact occurs, the two models are coupled via the contact stiffness and a different approach, outlined below, is used. If the contact stiffness is \( K_c \), the contact force will be given by

\[
F = K_c \left[ v(t) - w_R(t) - c \right]
\]  (1)

where \( v(t) \) and \( w_R(t) \) are the radial displacement of the pendulum and the ring and the initial radial clearance. It is assumed that the contact velocities are normal to the surfaces. The equations of motion for the ring can then be written in the following form

\[
[M] \ddot{\xi} + [C] \dot{\xi} + [K] \xi = \{Q(t)\}
\]  (2)

where \( \{Q(t)\} \) is the vector of generalised forces associated with the contact force and given by \( \{Q(t)\} = F \{ \cos \theta, \cos 2\theta, \ldots, \cos n\theta \}^T \). Introducing normal mode coordinates, \( \xi(t) \), for the ring [reference (3)], we may write \( \{q\} = [\Phi] \{\xi\} \) where \( [\Phi] \) is the modal matrix. Substituting into equation (2) and pre-multiplying by \( [\Phi]^T \) to transform to modal space, the \( r \)th modal space equation for the ring can be written as

\[
\ddot{\xi}_r + 2\gamma_r \dot{\xi}_r + \left( \omega_r^2 - \Omega_r^2 \right) \xi_r = \left[ [\Phi]^T \{Q(t)\} \right]_r
\]  (3)

where \( \omega_r \) and \( \gamma_r \) are the natural frequency and damping ratio of the \( r \)th mode. Similarly, for the single DOF rigid pendulum, which has natural frequency \( \Omega \), damping ratio \( \delta \) and moment of inertia \( I_Y \)

\[
\ddot{\xi} + 2\delta \Omega \dot{\xi} + \Omega^2 \xi = -\frac{F L^2}{I_Y}
\]  (4)

where \( \xi = L \phi_Y \). From equation (1), the contact force can therefore be written as

\[
F = K_c \left[ \xi - \sum_{r=1}^{N} \left[ [\Phi] \{\xi\} \right]_r - c \right]
\]  (5)

Introducing the state vector

\[
\{ u \} = \left\{ \{\xi\}, \{\dot{\xi}\}, \{\dot{\xi}\}, \{\dot{\xi}\} \right\}^T
\]

and eliminating the contact force, the modal equations of motion, equations (3) and (4) can be assembled in the form

\[
\{ \ddot{u} \} = \left[ \begin{array}{ccccc} A & B & C & D \\ E & F & G & H \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array} \right] \{ u \}
\]  (6)

These equations can be integrated numerically provided that the velocities and displacements of the two structures at the start of impact are known. The contact period will end when the force reduces to zero. The modal state vector, \( \{u\} \), at the end of the contact period gives the initial conditions for the subsequent period of non-contact motion.

The modal properties of the coupled system, which are the eigenvalues and eigenvectors of equation (6), have an important bearing on the duration of contact events, and will be discussed later.

3.3 Numerical Considerations

It is necessary to consider the computational time-steps to be used during both the non-contact and contact phases, along with the accuracy with which the start and end of contact need to be determined. During non-contact the required time-step depends on the frequency content of the motion and a value of 100 \( \mu s \) was found to be appropriate.

The time at which contact begins and ends is determined by the inequality \( v(t) - w_R(t) > c \). The inequality was tested at each time step and an iterative interpolation scheme was employed to find the actual start of the contact more accurately. Investigations were performed to determine an appropriate interpolation tolerance. For illustration, consider tolerances of 10\( \mu s \), 1\( \mu s \) and 0.1\( \mu s \). Results are presented for the case where the system is initially at rest and motion of the pendulum is initiated by a radial impulse in the OX direction. This leads to three closely spaced impacts between the pendulum and the ring, all at the same point (\( \theta = 0^\circ \)). The predicted force functions are shown in Figure 4.

With a tolerance of 10\( \mu s \), the predicted start and end of each force pulse is significantly different from zero, in either the compressive or the tensile sense. When the tolerance was reduced to 1\( \mu s \), the start and end of each force pulse are close to, but visibly different from, zero. With a tolerance of 0.1\( \mu s \), the start and end of each force pulse is sensibly zero. Smaller tolerances produced no discernible change and a value of 0.1\( \mu s \) was therefore used throughout.
The choice of time step to be used for numerical integration of the equations of motion during contact is governed by the need to have sufficient steps during one cycle of the highest mode of the coupled system. The natural frequency of this mode depends on the number of generalised coordinates used to model the ring. For the present system, a time-step of 1 µs was found to be satisfactory when up to 20 generalised coordinates were used for the ring.

3.4 The effect of the number of ring generalised coordinates on contact duration.

The purpose of the following discussion is to explain some model-dependent aspects of the predicted behaviour. Figure 5 shows the effect on the coupled natural frequencies of varying the number of generalised coordinates used to model the ring.

With N ring coordinates, there are N + 1 coupled modes (the extra degree of freedom being that of the rotor). Mode 1 is characterised by coupled rigid-body motion of the ring and rotor, the two members moving in phase with each other causing relatively little deformation of the contact spring. Since this mode depends on the rigid body behaviour of the ring, its frequency is independent of the number of flexible ring coordinates used.

For modes 2 to N, the main contribution to the rth coupled mode shape is found to be from ring coordinates r and r-1. If fewer than 6 coordinates are included, the motion of the ring and rotor at the point of contact is in-phase. For six or more coordinates, modes 6 and above exhibit out-of-phase movement at the point of contact. When such modes are excited during contact, there is significant compression of the contact spring, which in turn influences the contact behaviour. This is discussed later.

As an example, Figure 6 shows the mode shapes for modes 4, 11 and 12 for the model with 11 ring coordinates. Figure 7 shows the corresponding contributions of the generalised coordinates of the ring and the rotor (marked P on the axis). For mode 4, the largest contribution is from ring coordinate 4, with a smaller, negative contribution from coordinate 3.
the point of contact, the displacements of both the ring and rotor are positive and the mode shape of the ring exhibits the expected four lobes.

For mode 11, the largest contributions are from coordinates 11 and 10, and the mode shape has eleven lobes. At the point of contact, the displacement of the ring is negative while that for the rotor is positive. This results in a larger deformation of the spring than for mode 4.

Mode 12 (the highest frequency mode) has significant contributions from all coordinates, with coordinate 11 being the largest. At the point of contact, the displacement of the ring is large and negative while that for the rotor is positive. This again results in a large deformation of the spring.

Figure 8 shows the effect of the number of ring coordinates on the predicted contact duration.

Increasing the number of coordinates from 1 to 12 suggests that convergence of the contact duration might be expected at a value of about 0.075 ms. In this region, the behaviour of the system during contact is dominated by the highest frequency mode. As noted earlier, this mode has significant out-of-phase movement of the ring and the rotor at the point of contact. Excitation of this mode results in the compression and extension of the contact spring over approximately half a cycle. Once the spring returns to its unstrained length, separation occurs. As might be expected from this behaviour, the corresponding variation of the contact force (the 4-coordinate case in Figure 9) approximates to a half-sine wave.

As an example, consider the case of 4 ring coordinates for which the frequency of the highest mode is 4.5 kHz (Figure 5). The duration of half a cycle is 0.111 ms and the actual contact duration is found to be 0.112 ms. A similar correlation between the contact duration and the half periods of the highest frequency modes exists when from 1 to 12 ring coordinates are used. These are shown as triangles in Figure 8. With fewer than 6 coordinates included, the discrepancy is less than 2%. The discrepancy increases if more coordinates are used and is 33% with 12 coordinates. This occurs because it is found that the highest frequency mode alone no longer dominates the contact behaviour. With 12 coordinates, for example, modes 10, 11 and 12 make significant contributions in addition to mode 13. It will be seen from the frequencies in Figure 5 that these modes have half-periods which are shorter than the contact duration. As a result, they contribute extra oscillatory behaviour. This effect continues as more ring modes are included. With 15 ring coordinates for example, the contribution of these other modes prevents the contact spring from returning quickly to its unstrained length and thus delay separation. The presence of these additional oscillatory components can be clearly seen in the force plot in Figure 9 (15 coordinates).

There are thus two distinct regions created by the model. With fewer than 12 ring coordinates, the linear spring models the contact deformation as a local effect in which the global deformation of the ring is relatively small. When more coordinates are included, the deflection of the ring becomes similar to the compression of the spring and, in this case, it is doubtful that the continued use of the linear spring to model the local deformation remains valid. Further work, including experimental evidence, is required to clarify this.

Thus, if the ring is sufficiently stiff that its radial deflection is small in comparison with the local deformation at the point of
contact, the linear spring model provides a realistic approximation to the behaviour. This work forms part of a larger study of the rotor dynamics (1) and, while the detail of the individual impacts is important, the behaviour over many revolutions is of more significance for the overall integrity of the system. For the case investigated, the use of four ring coordinates was found to be sufficient for this purpose.

In other applications (sports equipment is an obvious example) in which the deflections of the impacting structures are comparable to any local contact deformation, many more coordinates would be required. This would be true particularly in cases involving single impacts.

4. CONCLUSIONS

The predicted behaviour of the system during contact has been shown to fall into two regimes, depending on the number of generalised coordinates used to model the elastic body.

When only a few ring coordinates are included (up to 12 in the case reported), it has been seen that the behaviour is dominated by the highest coupled mode. This depends on the contact model used; in this work, a linear spring with energy dissipation at the contact neglected. When a larger number of ring coordinates is included, the flexibility of the ring itself (expressed in terms of its higher coupled modes) plays an increasingly important role and when sufficient of these higher modes are present, a converged solution is obtained. If too few ring modes are used, a misleading prediction of the contact duration may be obtained.

Further work is required to validate the existing model experimentally and to extend the model for rotor dynamics applications by incorporating shaft rotation, contact friction, dissipation and gyroscopic effects.

REFERENCES

(2) K L Johnson, Contact Mechanics, Cambridge University Press, 1985
(3) L. Meirovitch, Elements of Vibration Analysis, McGraw-Hill, 1975

<table>
<thead>
<tr>
<th>Shaft radius $R$</th>
<th>Shaft length $L$</th>
<th>Shaft spring position $L_s$</th>
<th>Shaft spring stiffness $k_{SHAFT}$</th>
<th>Rotor radius $R_0$</th>
<th>Rotor length $L_D$</th>
<th>Rotor mass $M_D$</th>
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<td>10 mm</td>
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<td>Damping ratio $\gamma$</td>
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Table 1 System parameters