Optimal Placement of Actuators and Sensors for Vibration Active Control

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ABSTRACT

Optimal placement of actuators and sensors is an important issue in vibration active control. An index for joint consideration of controllability and observability based on singular values of Hankel matrix is described at first, then an objective function according to the index is defined for actuator and sensor placement. A genetic algorithm procedure (GA) is adopted to solve the optimization problem. A constrained mutation operation is proposed to avoid constraint violations in the genetic algorithm. The proposed optimal placement approach is applied to multiple input control of a space truss structure with closely-spaced modal frequencies. Computer simulations show that it is effective. The new approach can not only reach the global optimization, but also provide sub-optimal solutions which will be beneficial for the engineering applications.

NOMENCLATURE

|M| , |D| , |K| | Mass/damping/stiffness matrix
|C_d| , |C_v| | see Eq(1)
|Φ_f| , |Y_d| , |Y_v| | see Eq(2)
|A| , |B| , |C| | see Eq(5)
|F| , |f| | actuator influence matrix/force vector
[Z], [λ], [θ] | modal damping/frequency/eigenvector matrix
[w], [w_o] | controllability/observability gramian matrix
[h], [h_o] | hankel Matrix
[R] | gain matrix
[I] | identity matrix
{u}, {n} | physical response/modal coordinate vector
{r}, {y} | state/output vector
n | the dimension of the controlling system
ξ_i, Ω_i | the ith modal damping and frequency
σ_i | the ith singular value of Hankel matrix
p, q | desired number of actuators/sensors
m | number of controlled modes
J | objective function
N | the size of a population
α | constant

t | the number of candidate actuator locations
s | the number of candidate sensor locations
γ_i | value of the ith bit of a chromosome
fit_max | the largest fitness value in a population
fit_avg | the average fitness value of a population
γ_i | fitness value of the ith chromosome
p_i | probability for the ith chromosome to be chosen as a candidate parent
ε | convergence tolerance
i | ith
T | transpose
det( ) | determinant
( ) | vector
|| | absolute value

1 INTRODUCTION

There are some papers discussing exciter and sensor placement issue in modal testing. However, the optimal placement of actuators and sensors for vibration active control is more important because it relates to the controller gain and determines directly the performance of the controller system. Much work has been done to deal with the optimal actuator and sensor placement problem in vibration active control. In light of these papers, two aspects of the problem are intensively investigated: one is performance indices and the other is optimal methods.

Various performance indices are used in the optimal placement of actuators and sensors. Among a few of these, Viswanathan defines a degree of observability and controllability based on the area of ‘recovery region.’ Lim proposes an approach for placing actuators and sensors that is based on projecting eigenvectors into the intersection subspace of the controllability and observability subspaces for each sensor-actuator pair. Montgomery places actuators and sensors with the system failure consideration. However, the indices based on the observability and controllability are widely used because of their importance in controller design.

Of all the proposed indices, only the measures proposed by Maghami et al result in a single objective optimization problem instead of multiple objective one, in which a function of the singular values of the Hankel matrix is maximized that includes both measures of controllability and observability.

Optimal placement of actuators and sensors is essentially a
combining optimal problem. To date, some methods are developed to solve the combinatorial problem. Maghami and Joshi transform the discrete nature of actuator and sensor placement problem into a continuous nonlinear programming with the aid of functional approximation of the control forces and output measurement. Chen uses simulated annealing to place actuators in adaptive structures. Rao and Pan demonstrate the effectiveness of the genetic algorithm for the actuator placement in actively controlled structures.

Generic algorithms (GA), proposed by Holland, is a form of direct random search methods based on the 'survival of the fittest' and is radically different from the traditional optimal methods such as nonlinear programming. The GA is more likely to obtain the optimal solution because no gradient is used. Furthermore, the GA searches from a lot of points in the solution space, so it can provide some sub-optimal solutions besides the optimal one. When applying the GA to solve the problem of actuator and sensor placement, these solutions give physical insights into the placement problem and are beneficial for the engineering applications. As pointed out by Hajela, the GA has great promises to solve the combinatorial optimization problems.

When applying GA to solve the nonlinear optimal problem with constraints, a penalty-function formulation is generally used to treat the constraints. However, the penalty-function method is extremely sensitive to user-specified schedules of selecting penalty parameters. It biases the search toward the sub-optimal solutions and increases the number of function evaluations required to converge. Moreover, the chromosomes created by the reproduction strategies in reference will produce children which violate the constraints when the numbers of actuator and sensor are given.

In this paper, the reproduction strategies are modified so that all the chromosomes are satisfied the constraints. In section 2, the performance index for optimal actuator and sensor placement is described. The GA and its modification are reviewed in Section 3. Section 4 includes the applications of the GA to place actuators and sensors optimally on a space truss structure with closely-spaced modes. Finally, conclusions are given in section 5.

2 Performance Index

The mathematical model of a linear, time-invariant flexible space structures is given by

\[
\begin{align*}
\{M\} \{\ddot{u}\} + \{D\} \{\dot{u}\} + \{K\} \{u\} &= \{F\}\{f\} \\
\{y\} &= \{C_d\} \{\dot{u}\} + \{C_v\} \{\dot{v}\}
\end{align*}
\]  

(1)

where \([M], \{D\}, \{K\} \in \mathbb{R}^{n \times n}; \{F\} \in \mathbb{R}^{n \times p}; \{u\} \in \mathbb{R}^{n \times 1}, \text{ and} \{f\} \in \mathbb{R}^{n \times 1}; \{y\} \in \mathbb{R}^{m \times 1}; \{C_d\} \text{ and } \{C_v\} \text{ are } q_o \times n \text{ sensor influence matrices and different with the type of sensors used.}

Employing the transformation \(\{u\} = \{\phi\}\{\eta\}\), Eq.(1) can be written in the modal coordinates as

\[
\begin{align*}
\{\dot{\eta}\} + 2\{Z\} \{\Lambda\} \{\dot{\eta}\} + \{\Lambda\}^T \{\dot{\eta}\} &= \{\phi\} \{\dot{f}\} \\
\{\dot{y}\} &= \{C_d\} \{\phi\}\{\dot{\eta}\} + \{C_v\} \{\phi\}\{\dot{\eta}\} = \{Y_d\}{\dot{\eta}} + \{Y_v\}\{\dot{\eta}\}
\end{align*}
\]  

(2)

where \([Z] = \text{diag}\{\xi_i\}\) and \([\Lambda] = \text{diag}\{\omega_i\}\). \([\Phi]\) is a \(n \times m\) mode shape matrix normalized to yield a unity generalized mass matrix. Define a state vector

\[
\{x\} = \{\eta\}^T \{\dot{\eta}\}^T
\]  

(3)

Then an equivalent state-space representation for Eq.(2) becomes

\[
\begin{align*}
\{\dot{x}\} &= \{A\} \{x\} + \{B\}\{f\} \\
\{y\} &= \{C\} \{x\}
\end{align*}
\]  

(4)

where

\[
\begin{align*}
\{A\} &= \left[ \begin{array}{c} 0 \\ -[\Lambda]^T - 2[Z\Lambda] \end{array} \right] \\
\{B\} &= \left[ \begin{array}{c} 0 \\ [\phi]\{\dot{f}\} \end{array} \right] \\
\{C\} &= \left[ \begin{array}{c} [Y_d] \\ [Y_v] \end{array} \right]
\end{align*}
\]  

(5)

For linear, time-invariant systems, Controllability grammian \([W_c]\) can be obtained from the solution of a Lyapunov equation

\[
\{A\} \{W_c\} + \{W_c\} \{A\}^T = -[B][B]^T
\]  

(6)

Observability grammian \([W_o]\) can also be obtained from the solution of the following equation

\[
\{A\}^T \{W_o\} + \{W_o\} \{A\} = -[C][C]^T
\]  

(7)

The system is controllable only if the matrix \([W_c]\) is full rank and observable only if the matrix \([W_o]\) is full rank. The singular values of \([W_c]\) can be used as measures of controllability and those of \([W_o]\) as measures of observability. The actuators and sensors can be placed according to these measures, respectively. However, a joint consideration of actuator and sensor placement problem would have some advantages. To achieve the purpose, we apply similarity transformation to Eq.(4) and change the system into balance coordinates, then the controllability and observability grammians are equal and diagonal, i.e.,

\[
\{W_c\} = \{W_o\} = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{2m})
\]  

(8)

The singular values \(\sigma_1, \cdots, \sigma_{2m}\) are directly related to the matrices \([W_c]\) and \([W_o]\), so they can be used to measure the controllability and observability of the system. If one of the singular values is zero, the system has one or more modes which can not be controlled or observed. Furthermore, The determinant of \([W_c]\) and \([W_o]\) matrices represent the volume of region of controllability and observability, so the objective function can be defined as:

\[
J = \alpha \times \det(\{W_c\}) = \alpha \times \det(\{W_o\}) = \alpha \times \sigma_1 \times \cdots \times \sigma_{2m}
\]  

(9)

If one singular value is zero, \(J\) is also zero. It indicates that there are uncontrollable or unobservable modes existed in the control system.
3 Genetic Algorithm

In this section, the implementation of the GA for solving the actuator and sensor placement problem is described in detail. In the GA, there are three main operators: parent selection, crossover, and mutation. The full scheme is described as follows.

3.1 Coding of candidate actuator and sensor locations

In the GA, the search is conducted from a population of coded candidate actuator and sensor locations. In the work, each candidate actuator is coded as a bit binary number. When the number on a bit is 1, it indicates the location corresponding to the bit is equipped with an actuator. If the number on a bit is 0, then there is no actuator on the location. If there are t candidate actuator locations, they will be coded as t-bit binary string, so the s candidate sensor location coding. A chromosome is obtained by concatenating actuator location string and sensor location string to obtain a single string of ones and zeros which form a chromosome. If t=5, s=5, a coded chromosome is showed in Fig 1.

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

| 5-bit coding of actuators | 5-bit coding of sensors |

Fig 1: Coding of 5-sensor and 5-actuator candidate location problem

In Fig 1, the chromosome indicates that there are 2 actuators placed on the location 2,4, and 2 sensors on the location 1,5, respectively. N chromosomes will make up of a population. After coding, the objective function can be defined with the constraints as

\[
\begin{align*}
\max \quad J &= \alpha \times \det(\hat{W}_c) = \alpha \times \det(\hat{W}_s) = \alpha \times \sigma_1 \times \cdots \times \sigma_{2m} \\
\text{st.} \quad g_1 + g_2 + \cdots + g_t &= p_a \\
g_{t+1} + g_{t+2} + \cdots + g_{2m} &= q_s
\end{align*}
\]

(10)

3.2 Evaluation of objective function and chromosome fitness

Fitness is an important concept in the GA because it dominates the extent to which a chromosome can influence further generations. The values of the objective function are usually taken as the chromosome fitness values in an unconstrained optimal problem. While, the values of the objective function plus the penalty are often taken as fitness values in a constrained one. As described following, though the actuator and sensor placement problem in this work is a constrained one, all the children are forced to satisfy with the constraints. Thus, the values of the objective function are taken directly as chromosome fitness values.

To protect the GA from premature convergence and random search, the \(i^{th}\) chromosome fitness value is scaled as:

\[
f_i = e^{\lambda_i}
\]

(11)

GA is an iterative process, so the following

\[
|\text{Fit}_{\text{max}} - \text{Fit}_{\text{avg}}| \leq \varepsilon
\]

(12)

is used as the condition of convergence.

3.2 Selection

Selection is a strategy which chooses two chromosomes to crossover and favors the chromosomes with better fitness values. In Roulette wheel selection method, the probability that the \(i^{th}\) chromosome is selected as a parent is given by

\[
p_i = \frac{f_i}{\sum_{j=1}^{N} f_j}
\]

(13)

The larger the fitness value of the \(i^{th}\) chromosome, the larger the probability it be chosen as a parent, the greater influence it has to the further generation.

3.3 Crossover

When two parents have been selected, a biased coin is tossed to determine whether these two parents can produce children. In the work, a uniform crossover operator is introduced. Besides the selected two parents, a masking binary number is randomly produced with the length equal to the string length of a chromosome and is used with the operator. Let parent 1 link to child 1, and parent 2 to child 2, then the crossover can be described as follows. If a 1 on the \(i^{th}\) bit of the masking string, the value of the \(i^{th}\) bit of parent 1 is copied to that of child 1, and the value of the \(i^{th}\) bit of parent 2 to that of child 2. However, if a 0 on the \(i^{th}\) bit of the masking string, the value of the \(i^{th}\) bit of child 1 is given as that of parent 2, and the value of the \(i^{th}\) bit of child 2 is given as that of parent 1. The uniform crossover operator is illustrated in Fig 2, in which \(t=5, p_a=2, s=5, q_s=2\).

3.4 Mutation

Basic Mutation works on bit-bit basis with a low probability and it can improve the convergence performance.

Even though the parents are satisfied with the constraints before the crossover and basic mutation operation, it is very possible to produce children which violate the constraints as illustrated in Fig 2. The generated child 1 contains only one actuator and one sensor, however, child 2 is equipped with 3 actuators and one sensor. Both children violate the constraints. To make the produced children satisfy the constraints, a constrained mutation operator is developed.

For any newly generated child, the number of actuators \(p\) and sensors \(q\) in the child are checked. If \(p\) is larger than the desired actuators \(p_a\), a number of bits equal to \(p-p_a\) chosen randomly from the \(p\) bits are set to 0. On the other hand, if \(p<p_a\), a number of bits equal to \(p_a-p\) chosen randomly from all the bits corresponding to actuator locations but different from the \(p\) bits are set to 1. The
violations of sensors are dealt with as those of actuators. Fig 3 demonstrates the constrained mutation operator.

parent 1 1000100110
parent 2 1000101001
masking child 1 0000101100
child 2 1100100011
mutation child 1 0000101000
mutation child 2 1100100001

Fig 2: Uniform crossover and basic mutation operator

old child 1100100001
with p=3,q=1,p'=2,q'=2
random no. [p-p',q-q]=[9]
new child 1100100011

Fig 3: Constrained mutation operator

To apply the constrained mutation operator to all of the new produced infeasible children, all of the children in a population will satisfy with the constraints.

3.5 Population

In the GA, Fitmax and Fitavg may oscillate when the Fitavg is close to Fitmax. The steady-state GA is introduced to overcome the problem. In detail, when the N children are created, the N children and the N parents are sorted out in decreasing order. Then the first N chromosomes with better fitness values are chosen as the candidate parents. The method can make the Fitmax and Fitavg increase monotonically.

The algorithm is summarized as follows:

1. Initialize N chromosomes. Set the tolerance ε .
2. Compute the fitness values of the chromosomes.
3. Generate N children. Perform basic mutation along with the crossover procedure.
4. Apply the constrained mutation operator to the newly produced infeasible children.
5. Compute the fitness values of all the produced children. Sort out the N chromosomes with the larger fitness from the new children and parents as candidate parents.
6. If |Fitmax - Fitavg| < ε , stop; otherwise, go back to step 3 and continue.

4 Space Structure Example

The cantilevered, 7-bay space truss structure shown in Fig 4 is employed to demonstrate the proposed procedure. Each bay of the truss is a cube with the side dimension of 0.3m. The truss contains 96 truss members which are made of aluminum tubes with diameter of $\phi 6 \times 1 \text{mm}$. The modulus of elasticity and the mass density of the truss members are $7.27 \times 10^{10} \text{ N/m}^2$ and $3100 \text{kg/m}^3$, respectively. Each node is added with a centralized mass of 0.6kg. The nodes 1-4 are clamped, so the truss has 84 DOFs (3 DOFs per node) which can be deployed with actuators and sensors. The relationship between parts of nodes and DOFs (candidate actuator and sensor locations) is shown in Table 1. Using a general purpose finite element program, the natural frequencies are calculated and shown in Table 4 (indicated as uncontrolled modes). The first frequency is 11.44Hz, and is very close to the second frequency (11.62Hz). The bending modes are paired due to the symmetry of the truss.

4.1 Example 1

In example 1, the problem of placing 2 actuators and 2 sensors optimally is considered. Let $\alpha = 20000$. The iteration history of Fitmax and Fitavg is shown in Fig 5. At the beginning, Fitmax and Fitavg rise quickly. While they increase slowly near to convergence. It takes 46 generations to converge to the fitness value of 8.2685. The number of required objective evaluations is 4600, as opposed to 12152196 objective function evaluations required for the complete exhaustive search. The ratio is $3.79 \times 10^{-4}$ which indicates that the GA is a very efficient optimization method for the actuator and sensor placement problem.

The results of the GA are summarized in column two in Table 2.

<table>
<thead>
<tr>
<th>Node</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOF</td>
<td>73-75</td>
<td>76-78</td>
<td>79-81</td>
<td>82-84</td>
</tr>
</tbody>
</table>

* the DOFs arranged in x, y, z direction

The modified GA is implemented in the section. Probabilities for crossover and mutation operator for the GA are prescribed as 0.8 and 0.005 respectively, and set N=100. The tolerance $\varepsilon$ for convergence is selected as $10^{-5}$. Example 1 and 2 demonstrate the results of placing two actuators and two velocity sensors or 5 velocity sensors on the same 7-bay truss structure to control the first and second mode, respectively.
should be pointed out that the results are representative of several cases run in that the algorithm always returns the optimal solution. As the number of objective evaluations required for the complete exhaustive search is too large to carry out, we use a substitute method to get all good actuator and sensor configurations with the largest probability. Firstly the more important area of actuator and sensor locations is decided by employing the GA. Then exhaustive search is only conducted in the area. The results obtained by the method are listed in column one in Table 2. It is clear from Table 2 that the GA can identify some sub-optimal actuator and sensor configurations besides the best one. These sub-optimal actuator and sensor configurations give not only physical insights into the placement problem, but facilitate the designer to adjust the actuator and sensor positions as well. From the column two in Table 2, the location 73,76,77,80 is more important than other locations because the four locations appear in the 5 configurations with the largest fitness values.

**Table 2: Actuator and sensor configuration**

<table>
<thead>
<tr>
<th>Exhaustive search</th>
<th>GA</th>
<th>Exhaustive search</th>
<th>GA</th>
</tr>
</thead>
<tbody>
<tr>
<td>actuator</td>
<td>sensor</td>
<td>actuator</td>
<td>sensor</td>
</tr>
<tr>
<td>76,80</td>
<td>76,77</td>
<td>8.2390</td>
<td>76,77</td>
</tr>
<tr>
<td>76,77</td>
<td>76,80</td>
<td>8.2390</td>
<td>76,77</td>
</tr>
<tr>
<td>73,77</td>
<td>76,77</td>
<td>8.2390</td>
<td>73,77</td>
</tr>
<tr>
<td>76,77</td>
<td>73,77</td>
<td>8.2390</td>
<td>76,77</td>
</tr>
<tr>
<td>76,77</td>
<td>76,77</td>
<td>8.2685</td>
<td>76,77</td>
</tr>
</tbody>
</table>

4.2 Example 2

In example 2, 2 actuators and 5 sensors are placed on the truss structure. Let $\alpha = 1000$. Fig 6 gives the convergence histories, and from which, conclusions similar to those in Example 1 may be derived. It takes 50 generations to converge to the fitness value of 13.1592, and the number of required objective evaluations is 5000. While the complete exhaustive search requires $1.08 \times 10^{11}$ objective evaluations. The ratio is $4.65 \times 10^{-8}$. It is clear that the objective evaluations required by the GA increases linearly, however, those required by complete exhaustive search rise factorically.

**Fig 6: Iteration history of 2 actuator and 5 sensor placement problem**

<Fig 6: Iteration history of 2 actuator and 5 sensor placement problem (- Fit$_{max}$ ----- Fit$_{avg}$)

**Table 3: Actuator and sensor configuration**

<table>
<thead>
<tr>
<th>Exhaustive search</th>
<th>GA</th>
</tr>
</thead>
<tbody>
<tr>
<td>actuator</td>
<td>sensor</td>
</tr>
<tr>
<td>73,77</td>
<td>73,76,77,79,80</td>
</tr>
<tr>
<td>76,77</td>
<td>73,76,77,80,82</td>
</tr>
<tr>
<td>76,77</td>
<td>73,74,76,77,80</td>
</tr>
<tr>
<td>76,77</td>
<td>73,76,77,79,80</td>
</tr>
<tr>
<td>76,77</td>
<td>73,76,77,80,83</td>
</tr>
</tbody>
</table>

The results determined by exhaustive search as described in example 1 and by the GA approach are given in Table 3. It is shown that the locations 73,76,77,80 are more important actuator locations, while the locations 73,74,76,77,80,82,83 are more important sensor locations for the example. These locations are at the free end of the truss as expected.

4.3 Control demonstration

The direct velocity feedback strategy is employed to demonstrate the performance of the optimal actuator and sensor locations obtained by GA, i.e.,

$$\{f\} = -[R]\{x\}$$

Let gain matrix $[R] = 100 \times [I]$. 2 actuators are placed on location 76,77, and 2 sensors are also placed on location 76,77. The modal parameters controlled are given in Table 4. After Controlled, the damping ratios of the first and second mode increase to 12.28%,
11.51%, respectively. These two modes are almost damped out. The damping ratios of other three modes also rise because of spillover. The responses of location 76, 77, i.e., the x, y direction at node 30, are depicted in Fig 7, where the controlled responses decrease much more quickly than those uncontrolled.

Table 4: the uncontrolled/controlled mode parameters

<table>
<thead>
<tr>
<th>No.</th>
<th>uncontrolled modes</th>
<th>controlled modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>frequency (Hz)</td>
<td>damping ratio(%)</td>
</tr>
<tr>
<td></td>
<td>11.44</td>
<td>0.40</td>
</tr>
<tr>
<td>1</td>
<td>11.62</td>
<td>0.40</td>
</tr>
<tr>
<td>2</td>
<td>31.37</td>
<td>1.00</td>
</tr>
<tr>
<td>3</td>
<td>51.84</td>
<td>0.80</td>
</tr>
<tr>
<td>4</td>
<td>54.71</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>11.38</td>
<td>11.38</td>
</tr>
<tr>
<td></td>
<td>11.55</td>
<td>11.51</td>
</tr>
<tr>
<td></td>
<td>31.30</td>
<td>2.99</td>
</tr>
<tr>
<td></td>
<td>51.80</td>
<td>1.75</td>
</tr>
<tr>
<td></td>
<td>54.64</td>
<td>1.90</td>
</tr>
</tbody>
</table>

![Graph](image1)

![Graph](image2)

Fig 7: The displacement response

(a) the response in x direction at the node 30

(b) the response in y direction at the node 30

5 Conclusions

The Genetic Algorithm (GA) has been successfully applied to solve the actuator and sensor placement problem for vibration active control of a space truss with closely-spaced modes. A constrained mutation operator is proposed to make the children satisfy the constraints. The GA is much more efficient than the exhaustive search. Furthermore, the new approach can not only reach the global optimization, but also provide sub-optimal solutions which will be beneficial for the engineering applications.

6 ACKNOWLEDGMENT

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7 REFERENCES