CONDITIONED FREQUENCY RESPONSE ESTIMATORS FOR NONLINEAR SYSTEMS

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ABSTRACT

Conventional frequency response estimation methods such as the "H_1" and "H_2" methods often yield measured frequency response functions which are contaminated by the presence of nonlinearities and hence make it difficult to extract underlying linear system properties. To overcome this deficiency, a new spectral approach for identifying multi-degree-of-freedom nonlinear systems excited by Gaussian excitations is introduced. Conditioned "H_4" and "H_2" frequency response estimates now yield the underlying linear properties without contaminating effects from the nonlinearities. The new spectral approach is successfully tested on several example systems which include a 3-degree-of-freedom system with an asymmetric nonlinearity, a 3-degree-of-freedom system with distributed nonlinearities and a 5-degree-of-freedom system with multiple nonlinearities and multiple excitations.

NOMENCLATURE

Bold characters indicate matrices and vectors.
A coefficient matrix of nonlinear function vectors
B linear dynamic stiffness matrix
C linear damping coefficient matrix
f elastic force
f^t excitation vector of Gaussian time history
F spectra of f
F_1[,] Fourier transform
G single-sided cross-spectral density matrix
H linear dynamic compliance matrix
i \sqrt{-1}
k linear stiffness coefficient
K linear stiffness coefficient matrix
M mass matrix
N dimension of system
n number of types of nonlinearities

PSD power spectral density
t time
x generalized displacement vector
X spectra of x
y nonlinear function vector
Y Spectra of y
\alpha coefficient of quadratic nonlinear stiffness terms
\beta coefficient of cubic nonlinear stiffness terms
\gamma coefficient of fifth-order nonlinear stiffness terms
\omega frequency

1. INTRODUCTION

The properties of multi-degree-of-freedom linear systems are typically identified using time or frequency domain modal parameter estimation techniques [1]. For the frequency domain techniques, the algorithms extract modal parameters from measured frequency response functions in the presence of uncorrelated noise using conventional "H_1" or "H_2" frequency response estimation methods [2]. However, if the system under identification also possesses nonlinearities, conventional methods often yield contaminated frequency response functions from which accurate modal parameters cannot be determined [3]. Such conventional methods are also incapable of identifying the nonlinearities.

To accommodate for the presence of nonlinearities, several researchers have developed methods to improve frequency domain analysis of nonlinear systems. For example, the functional Volterra series approach for estimating higher order frequency response functions of nonlinear systems has gained recognition [4]. This method has been used to estimate first and second order frequency response functions of a nonlinear beam subjected to random excitation [5], where curve fitting techniques were used for parametric estimation of an analytical model. Other higher
order spectral techniques have also been employed for the analysis of nonlinear systems [6]. For instance, the bi-coherence function has been used to detect the second order nonlinear behavior present in a system [7]. Also, the sub-harmonic responses of a high speed rotor have been studied using bi-spectral and tri-spectral techniques [8]. An alternative approach has recently been developed by Bendat et al. [9, 10] for single-degree-of-freedom nonlinear systems which is based on a “reverse path” system model. This approach has been found to be computationally less intensive than the higher order spectral approaches. As a result, an enhanced multi-degree-of-freedom spectral approach based on the “reverse path” system model has been developed [11, 12] and is introduced in this paper. Computational results are given to illustrate the performance for several nonlinear systems.

2. PROBLEM FORMULATION

The equations of motion of a discrete vibration system of dimension N with localized nonlinear springs and dampers can be described by the following set of coupled nonlinear differential equations:

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + \sum_{j=1}^{n} A_i y_j(t) = f(t) \]  

where \( A_i \) contain the coefficients of the nonlinear restoring force terms. From a system identification perspective, it is assumed that the types of the nonlinearities and their physical locations are known. Therefore, the \( n \) nonlinear function vectors \( y_j(t) \) can be calculated. This assumption renders limitations on the practical use of this method since various types of nonlinearities at each location are not always known. However, it should be noted that this restriction is currently true for any identification scheme for nonlinear systems. Reference [11] contains additional research which has been conducted to investigate this limitation along with some possible solutions.

Consider several multi-degree-of-freedom nonlinear systems as illustrated in Figure 1. The first example as shown in Figure 1(a) possesses an asymmetric quadratic-cubic nonlinear stiffness element which exists between the second and third masses:

\[ f_{23}^1(t) = k_2(x_2(t) - x_3(t)) + \alpha_2(x_2(t) - x_3(t))^2 + \beta_2(x_2(t) - x_3(t))^3 \]  

Assuming that the form of the nonlinear elastic force \( f_{23}^1(t) \) is known, \( y_1(t) \) and \( y_2(t) \) take the following form:

\[ y_1(t) = y_3(t) = (x_2(t) - x_3(t))^2 \]  
\[ y_2(t) = (x_2(t) - x_3(t))^3 \]  

Since two types of nonlinearities (quadratic and cubic) exist at a single junction, \( y_1(t) \) and \( y_2(t) \) both contain the same relative displacements. Example II of Figure 1(b) has distributed cubic stiffness nonlinearities at every junction, therefore:

\[ f_{13}^2(t) = k_3(x_1(t) - x_3(t)) + \beta_3(x_1(t) - x_3(t))^3, \]
\[ f_{33}^2(t) = k_5(x_3(t) - x_5(t)) + \beta_5(x_3(t) - x_5(t))^5, \]
\[ y_1(t) = (x_2(t) - x_3(t))^3, y_2(t) = (x_2(t) - x_3(t))^3, y_3(t) = (x_2(t) - x_3(t))^3 \]  

Here a single type of nonlinearity exists at three junctions. Therefore, \( y_1(t) \) is a 3 by 1 column vector. Example III of Figure 1(c) is composed of a cubic nonlinear stiffness element between the second and third masses and an asymmetric nonlinear stiffness element described by a quadratic and fifth order term between the third and fifth masses. Therefore,

\[ f_{23}^3(t) = k_3(x_2(t) - x_3(t)) + \beta_3(x_2(t) - x_3(t))^3, \]
\[ f_{35}^3(t) = k_6(x_5(t) - x_3(t)) + \beta_6(x_5(t) - x_3(t))^5, \]
\[ y_1(t) = (x_2(t) - x_3(t))^3, y_2(t) = (x_3(t) - x_5(t))^2, y_3(t) = (x_5(t) - x_3(t))^5 \]  

Gaussian excitation \( f(t) \) is applied to mass 1 of Example I and II and masses 1 and 4 of Example III. Assuming that these excitations and the responses at the discrete locations are measurable, the problem is to identify the underlying modal parameters of these examples using the measured quantities without influence from the nonlinearities. Taking the Fourier transform \( F(\omega) \) of equation (1):

\[ B(\omega)X(\omega) + \sum_{j=1}^{n} A_j Y_j(\omega) = F(\omega) \]
\[ X(\omega) = F\{x(t)\}, Y_j(\omega) = F\{y_j(t)\}, F(\omega) = F\{f(t)\} \]  
\[ B(\omega) = -\omega^2M + i\omega C + K \]

The frequency domain system model, equation (6a), is composed of a linear dynamic stiffness matrix \( B(\omega) \), and terms representing the nonlinear elastic forces \( A_i Y_i(\omega) \). Using frequency domain based higher dimensional modal parameter estimation techniques [1], the modal parameters are extracted from the linear dynamic compliance matrix \( H(\omega) = B(\omega)^{-1} \). Two common methods for estimating the
dynamic compliance matrix (i.e. “H,” and “H” frequency response estimation methods [2]) can be applied directly to multiple-input/multiple-output data from a nonlinear system excited by Gaussian random excitation. However, effects from the presence of the nonlinear elastic forces \( A_i Y_i(\omega) \) can corrupt the underlying linear characteristics of the response causing non-Gaussian output and resulting in estimated dynamic compliance functions which often lead to erroneous results from modal parameter estimation. These nonlinear effects are illustrated using numerically simulated data from the example systems. A 5th order Runge-Kutta Fehlberg numerical integration method is used to calculate the response data. The time steps are held constant so that the Fourier transform can be applied to the data. Also, high frequency numerical simulation errors are minimized by choosing a Nyquist frequency eight times greater than the frequency range of interest.

One might argue that an improved estimate of the linear dynamic compliance functions could simply be obtained by exciting the systems at lower excitation levels, hence minimizing the nonlinear effects. However, reducing excitation levels to minimize the effects of nonlinearities makes the concurrent identification of the nonlinearities even more difficult. Also, for massive structures, large excitation levels may in fact be necessary in order to produce measurable responses at all of the desired output locations. Finally, nonlinear structures should be identified using excitation levels comparable to those experienced under real conditions (which may entail large excitations).

To address these issues, a multi-degree-of-freedom “reverse path” method is introduced and the examples of Figure 1 are used to illustrate its potential. This method starts with a “reverse path” model as discussed in the following section.

3. CONDITIONED “REVERSE PATH” FORMULATION

The concept of a “reverse path” model for single-degree-of-freedom systems is adapted from the works of Bendat et al. [9-10] but it is generalized here for the application to multi-degree-of-freedom systems. A multi-degree-of-freedom “reverse path” model [12] as shown in Figure 3(a) is derived by re-arranging equation (6a) with \( F(\omega) \) as the output and \( X(\omega) \) and \( Y_i(\omega) \) as the inputs to the model:

\[
F(\omega) = B(\omega)X(\omega) + \sum_{j=1}^{n} A_j Y_j(\omega) \quad (7)
\]

Using spectral conditioning technique [12], an equivalent conditioned “reverse path” model can be derived and Figure 3(a) can be redrawn with \( n + 1 \) uncorrelated input vectors as shown in Figure 3(b) where \( Y_{i(1:1)} \) and \( X_{i(1:1)} \) are mutually uncorrelated (the subscript \((-1:\ldots)\) indicates a vector uncorrelated with \( Y_1 \) through \( Y_{i-1} \)). Comparing Figures 3(a) and 3(b), notice that the coefficient matrices between each \( Y_{i(1:1)} \) and \( F_{i(1:1)} \) are not the original coefficient matrices \( A_i \) between \( Y_i \) and \( F \). This alteration is necessary in order for the overall model output \( F \) to remain unchanged. The
Figure 2. Linear dynamic compliance estimates. Key: o o o true linear dynamic compliance function, --- conventional “$H_1$” estimate, — conditioned “$H_{2c}$” estimate. (a) Example I. (b) Example II. (c) Example III.
original coefficient matrices are recovered once $B$ is identified as covered in reference [12]. However, the path between $X_{(-1:n)}$ and $F_{(-1:n)}$ remains unchanged. This path is the linear dynamic stiffness matrix $B$ and its input and output vectors are uncorrelated with all of the spectra of the nonlinear function vectors. Therefore, the underlying linear system can be identified without any corruption from the nonlinearities.

Since linear techniques (i.e. modal parameter estimation techniques) normally involve the dynamic compliance matrix $H$, and not the dynamic stiffness matrix $B$, identification of the linear path is conducted by re-reversing the flow of the linear path as illustrated in Figure 3(c). Now, any of the conventional frequency response estimation methods can be modified to estimate $H$. For example, the conditioned “$H_{c1}$” and “$H_{c2}$” estimates of the linear dynamic compliance matrix are as follows:

conditioned “$H_{c1}$” estimate: $H^T = G_{FF(-1:n)}^{-1} G_{FX(-1:n)}$  
(8a,b)

conditioned “$H_{c2}$” estimate: $H^T = G_{XF(-1:n)}^{-1} G_{XX(-1:n)}$

where the calculation of the conditioned PSD matrices $G_{XX(-1:n)}$, $G_{FF(-1:n)}$, and $G_{XF(-1:n)}$ is discussed in [12]. Once the conditioned frequency response functions have been estimated, modal analysis techniques can be used to extract natural frequencies, damping ratios and mode shapes without influence of the nonlinearities. Other modal indicators can also be used to evaluate the number of modes in the frequency range [1].

4. RESULTS

To illustrate the performance of the conditioned multi-degree-of-freedom “reverse path” approach, the simulated data used in section 2 is also be used here so that direct comparisons can be made between the conventional “$H_1$” estimates and the conditioned “$H_{c2}$” estimates.

Example I, consisting of the asymmetric nonlinearity is first considered. Since two different nonlinear function vectors are present (equation (3a,b)), the measured linear dynamic compliance functions are determined from the following conditioned “$H_{c2}$” estimate

$$H^T = G_{XF(-1:n)}^{-1} G_{XX(-1:n)}$$  
(9)

Since a single excitation is applied to mass 1 (i.e. $F = F_1$), only the first column of $H$ is identified. Also $G_{XF(-1:n)}$ is a 3 by 1 column vector, therefore equation (9) is solved in a least squares sense. A sample is shown in Figure 2(a). Notice, a considerable improvement has been made when comparing the conditioned “$H_{c2}$” estimate with the conventional “$H_1$” estimate.

Example II is next considered for identification where a single nonlinear function vector exists (equation (4d)). Therefore the estimated linear dynamic compliance functions are determined from the following conditioned “$H_{c2}$” estimate

$$H^T = G_{XF(-1:n)}^{-1} G_{XX(-1:n)}$$  
(10)

As with Example I, a least squares estimation is used for the solution of equation (10) since the measured PSD matrix $G_{XF(-1:n)}$ is a 3 by 1 column vector. A sample estimate is shown in Figure 2(b). As with Example I, an improved estimate of the linear dynamic compliance function is
obtained. As opposed to the "H_1" estimate for this example, all three modes of the conditioned "H_2" estimate match well with the actual underlying linear system modes.

Example III, the 5-degree-of-freedom system, is finally considered. Three nonlinear function vectors exist (equation (5c-e)), therefore the conditioned "H_3" estimate is

\[ H^T = G^{-1}_{XF(-13)} G_{XX(-13)} \]  

(11)

A sample estimate is illustrated in Figure 2(c). Comparing the conventional and conditioned estimate shows a considerable improvement in estimating this linear dynamic compliance function using equation (11).

These preliminary results illustrate improved estimation of the linear dynamic compliance functions of Examples I, II and III using the "reverse path" system approach. Apply modal parameter estimation techniques to the "H_3" estimated dynamic compliance functions from these examples would result in much more accurate parameters than those obtained using the conventional "H_1" estimates.

5. CONCLUSION

It has been shown in this paper that conventional frequency response estimation methods such as the "H_1" and "H_2" estimates are often inadequate for accurately estimating the linear dynamic compliance functions of multi-degree-of-freedom nonlinear systems when excited by Gaussian excitations. Therefore, a new spectral approach has been developed based on a "reverse path" formulation as available in the literature for single-degree-of-freedom nonlinear systems [9, 10], with emphasis on the development for application to multi-degree-of-freedom systems [11, 12]. With new formulation, conditioned "H_3" and "H_4" estimates of linear dynamic compliance functions can now be obtained which drastically reduce, or even eliminate in some cases, the contamination introduced by nonlinearities. This allows for the identification of the modal parameters of the underlying linear system without any undue influences caused by nonlinearities.

This new spectral approach has been tested on three example systems with polynomial nonlinearities. These systems were excited by Gaussian excitations applied at either one or two locations. The multiple-input/multiple-output data from these systems have been successfully used and the results illustrate benefits of this approach. However, further refinements and tools are necessary before the method can be applied to the measured input/output data of "real" nonlinear systems. For instance, modifications need to be made to accommodate for uncorrelated measurement noise. Reference [11] introduces coherence functions which can be used as a means to determine the validity of conditioned spectral estimates in the presence of measurement noise. These coherence functions can also aid in determining the validity of the models used for describing the nonlinearities.

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