TIME DOMAIN IDENTIFICATION PROCEDURE FOR FINITE ELEMENT MODELS BASED ON MODAL LEAST SQUARES

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ABSTRACT. The identification of finite element models is in general a difficult task, since severe numerical problems like the computational speed or divergence can be the consequence of the large number of DOFs of the model. This paper presents a solution of this problem for midsized FE-models within a time domain identification. The approach utilizes robust least squares estimators and modal analysis as tools to avoid the numerical problems associated with the model size. The identification of local damping models is made possible by assuming nonclassically damped structures throughout the computational procedures, implying the solution of the complex eigenvalue problem. For this purpose a subspace iteration method is adapted, which can be combined advantageously with the proposed iterative algorithm.

NOMENCLATURE

- correction factor of damping matrix
- damping matrix
- excitation vector
- identity matrix
- objective function
- correction factor of stiffness matrix
- stiffness matrix
- correction factor of mass matrix
- mass matrix
- transformation matrix
- displacement vector
- velocity vector
- acceleration vector
- state vector
- measurement vector
- vector of modal coordinates
- number of modal coordinates
- number of degrees of freedom
- number of damping correction factors
- number of stiffness correction factors
- number of mass correction factors
- number of measured DOFs
- total number of correction factors
- number of state variables
- number of measured time points

1. INTRODUCTION

Finite element (FE) models are in various engineering applications a common and well established tool to describe the dynamical behaviour of mechanical systems. Nevertheless, quite often components of the structural matrices are not well known. These components can be e.g. local damping elements or support stiffnesses. System identification methods are in this situation a very helpful tool to evaluate the parameters of these elements such that the model response is close to measurements from the real system. On the other hand severe numerical problems like divergence and the computational effort are
curing when FE-models are treated directly in system identification methods. Moreover the number of measurements is due to technical or cost reasons often strongly restricted, i.e. many degrees of freedom of the model are not monitored. Frequency domain methods are already published which are treating these problems (see e.g. [1],[3]). This is not the case for time domain identification, where numerical examples are generally restricted to a relatively small number of degrees of freedom (see e.g. [5],[6],[7]).

In this paper a time domain identification approach is presented which is addressing especially the identification of FE-models [4]. It is based on robust and simple least squares estimators and as a key for avoiding the numerical difficulties associated with the size of FE-models, the procedure is formulated in a subset of modal coordinates of an a priori model. The approach consists of two steps, which are repeated iteratively. In the first one state estimation is performed, i.e. based on the incomplete measurements a completed state vector is evaluated. Then in the second step the system parameters are determined based on this completed state information. Since in state as well as in parameter estimation only a linear equation system formulated in modal coordinates must be solved, the numerical effort is strongly restricted and a nonlinear optimization problem (see e.g. [5]) can be avoided.

In each iteration step of the algorithm the a priori model and the modal decomposition is updated. Hence, for each step the eigenvalue problem must be solved. For this purpose a subspace iteration method (see e.g. [2]) adapted to the iterative identification scheme is utilized to reduce drastically the numerical effort and to make the suggested approach based on modal decomposition numerically attractive. The algorithm allows the identification of local damping effects, since nonclassically damped structures are assumed. Therefore the modal decomposition of the structural matrices is performed by solving the complex eigenvalue problem.

A numerical example in form of a FE-model of a crane structure with 1692 DOF is given. The damping parameter of three local dampers and five stiffness parameters are identified.

2. STATEMENT OF THE PROBLEM

Linear FE-models of the form

$$\dot{M} \ddot{u} + C \dot{u} + Ku = g(t)$$

are considered in this paper. The matrices $M$, $C$ and $K$ are the mass, damping and stiffness matrices with dimensions $[n \times n]$ respectively. The vectors $u$ and $g(t)$ with dimension $[n \times 1]$ contain the displacement and deterministic excitation. Some submatrices $M_i$, $C_i$ and $K_i$ of the system matrices are considered to be uncertain and they are premultiplied by factors $m_i$, $c_i$ and $k_i$. These factors can be physical parameters like e.g. the youngs modulus or simply global correcting factors of the submatrices. In the presented approach it is assumed, that initial or a priori values of these factors (index 0) are known and that corrections $\Delta$ are identified. Hence, the system matrices are assumed to be linear combinations in form of

$$M = M_0 + \sum_{i=1}^{n_m} (m_{i,0} + \Delta m_i)M_i$$

$$C = C_0 + \sum_{i=1}^{n_c} (c_{i,0} + \Delta c_i)C_i$$

$$K = K_0 + \sum_{i=1}^{n_k} (k_{i,0} + \Delta k_i)K_i$$

The system matrices of the so called a priori model $M'$, $C'$ and $K'$ are determined with the initial values of the correcting factors and the not corrected parts $M_0$, $C_0$ and $K_0$ of the system matrices:

$$M = M' + \sum_{i=1}^{n_m} \Delta m_i M_i$$

$$C = C' + \sum_{i=1}^{n_c} \Delta c_i C_i$$

$$K = K' + \sum_{i=1}^{n_k} \Delta k_i K_i$$

In the presented iterative approach the system matrices of the priori model are updated in each iteration loop, i.e. the identified corrections are used to recalculate the initial values of correcting factors for the next step. The relation between the measurements $y(t_i)$ and the state vector of the model $x(t_i)$ is described by

$$y(t_i) = T x(t_i) \quad , \quad 1 \leq i \leq n_t$$
$T$ is a linear transformation matrix of dimension $[n_{me} \times n_{st}]$. $n_{st}$ is the number of state variables, $n_{me}$ the number of measured states and $n_t$ the number of discrete measured time points.

3. MODAL LEAST SQUARES PROCEDURE (MLSQ-ESTIMATOR)

The suggested approach is divided into two estimation steps in each iteration loop. Especially in case of FE-Models not all degrees of freedom can be observed. Therefore in the first step an estimate for the complete state vector, i.e. all displacements and velocities of the model is calculated (see section 3.2). Based on this completed state vector in the second estimation step the corrections $A$ in eq. (3) are identified (see section 3.3). The MLSQ-approach is formulated in both estimation steps in modal coordinates of the a priori model (see eq. (3)), i.e. an eigenvalue problem of this model is solved and then it is decomposed (see section 3.1). The numerical solution of the eigenvalue problem is described in section 3.4.

3.1 Modal Decomposition of the a priori model

A state space formulation of the form

$$
\begin{pmatrix}
C' & M' \\
M' & 0
\end{pmatrix}
\dot{x} = \begin{pmatrix}
-K' & 0 \\
0 & M'
\end{pmatrix} x = \begin{pmatrix}
g(t) \\
0
\end{pmatrix} = \begin{pmatrix}
c(t)
\end{pmatrix}
$$

(5)

with the state vector $x$

$$
x^T = (u^T, \dot{u}^T)
$$

(6)

is established in utilizing eq. (1) and the system matrices of the a priori model in eq. (3), i.e. the corrections $A$ are set to zero in this equation. Then the symmetric eigenvalue problem

$$
\begin{pmatrix}
C' & M' \\
M' & 0
\end{pmatrix} \Phi A - \begin{pmatrix}
-K' & 0 \\
0 & M'
\end{pmatrix} \Phi = 0
$$

(7)

is solved. The results are the modal matrix $\Phi$ and the diagonal matrix of eigenvalues $A$. Modal coordinates $z$ with

$$
z = \Phi z
$$

(8)

can be introduced to decouple eq. (5). The decomposed form can be solved analytically. The set of $j = 1, \ldots, n_{st}$ solutions reads

$$
z_j(t_{i+1}) = e^{\lambda_j(t_{i+1}-t_i)}z_j(t_i) + \int_{t_i}^{t_{i+1}} e^{\lambda_j(t_{i+1}-\tau)} \sum_{k=1}^{n_{me}} \Phi_{k,j} c_k(\tau) d\tau
$$

(9)

From these equations the subset of $m$ members is selected, which occurs in the considered frequency range. The measurement equation is described also in modal coordinates. This form is determined by introducing eq. (8) in eq. (4):

$$
y(t_i) = T\Phi z(t_i)
$$

(10)

3.2 State Estimation

In this step the modal coordinates $z(t_i)$ for all time points $t_i$ are estimated which fit the measurements in eq. (10) and the modal solutions in eq. (9). The optimal solution for all $z(t_i)$ is calculated by minimizing an objective function $J$ which reads

$$
J = \sum_{i=1}^{n_t-1} e_{1,i}^* e_{1,i} + \sum_{i=1}^{n_t} e_{2,i}^* e_{2,i}.
$$

(11)

The symbol $\ast$ denotes the complex conjugate. The vector $e_{1,i}$ contains $m$ residuals of the selected $m$ modal solutions

$$
e_{1,i,j} = z_j(t_{i+1}) - e^{\lambda_j(t_{i+1}-t_i)}z_j(t_i) - \int_{t_i}^{t_{i+1}} e^{\lambda_j(t_{i+1}-\tau)} \sum_{k=1}^{n_{me}} \Phi_{k,j} c_k(\tau) d\tau
$$

(12)

The vector $e_{2,i}$ holds the residuals of the measurement equations

$$
e_{2,i} = \Psi^T T^T y(t_i) - \Psi^T T^T \Phi z(t_i)
$$

(13)

The abbreviation $\Psi$ means

$$
\Psi = \begin{pmatrix}
C' & M' \\
M' & 0
\end{pmatrix}
$$

(14)

The result of the minimization of the objective function $J$ is a linear equation system:

$$
U \begin{pmatrix}
z(t_1) \\
\vdots \\
z(t_{n_t})
\end{pmatrix} = u
$$

(15)
The matrix $\mathbf{U}$ has a band structure with band width $m+1$ (detailed form see Appendix). The modal coordinates $\mathbf{z}(t_i)$ can be transformed to real space with eq. (8).

### 3.3 Parameter Estimation

The corrections $A$ of the a priori correcting factors or parameters (see eq. (3)) are combined in a vector

$$ \Delta \mathbf{\theta} = (\Delta m_1, \ldots, \Delta m_m, \Delta c_1, \ldots, \Delta c_n, \Delta k_1, \ldots, \Delta k_n) $$

and are called "a priori" because these factors are determined on the basis of the a priori model. After introducing eq. (3) into eq. (1), a state space description similar to eq. (5) can be established

$$ \begin{pmatrix} \mathbf{C}' & \mathbf{M}' \\ \mathbf{M}' & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\mathbf{K}' & 0 \\ 0 & \mathbf{M}' \end{pmatrix} \mathbf{x} = \mathbf{c}(t) + \mathbf{D}(t) \Delta \mathbf{\theta} $$

where

$$ \mathbf{D}(t) = \begin{pmatrix} -\mathbf{H}_1(t) & -\mathbf{H}_2(t) & -\mathbf{H}_3(t) \\ 0 & 0 & 0 \end{pmatrix} $$

and

$$ \begin{align*}
\mathbf{H}_1(t) &= (\mathbf{M}_1 \mathbf{u}(t), \ldots, \mathbf{M}_m \mathbf{u}(t)) \\
\mathbf{H}_2(t) &= (\mathbf{C}_1 \mathbf{u}(t), \ldots, \mathbf{C}_n \mathbf{u}(t)) \\
\mathbf{H}_3(t) &= (\mathbf{K}_1 \mathbf{u}(t), \ldots, \mathbf{K}_n \mathbf{u}(t))
\end{align*} $$

Since all states are known from the first step of the approach only the vector $\Delta \mathbf{\theta}$ is not known in eq. (17). To find an optimal solution for the $\Delta \mathbf{\theta}$ at all time points the objective function $J$ defined by

$$ J = \sum_{i=1}^{n-1} \mathbf{e}_i^T \mathbf{e}_i $$

is minimized. The vector $\mathbf{e}_i$ represents the errors of the modal solution of eq. (17). It contains the modal solution error of the system equation as in the residuals $\mathbf{e}_{1,i}$ (see eq. (12)) and an additional term including the desired parameter corrections:

$$ \mathbf{e}_{1,i,j} = \mathbf{e}_{1,i,j} - \sum_{l=1}^{n_{\text{param}}} V_{i,j,l} \Delta \theta_l $$

where

$$ V_{i,j,l} = \int_{t_{i}}^{t_{i+1}} e^{\lambda_j(t_{i+1}-\tau)} \sum_{k=1}^{n_{\text{stat}}} \Phi_{k,j} D_{k,l} $$

Since the left hand side of eq. (17) contains the system matrices of the a priori model, the eigensolution is the same as in the state estimation step. The analytical result of the minimization of $J$ is again a linear equation system

$$ \mathbf{V}^T \mathbf{r} = \mathbf{V}^T \mathbf{\Delta \theta} $$

which is the desired solution equation for $\Delta \mathbf{\theta}$. The matrix $\mathbf{V}$ and the vector $\mathbf{r}$ are structurized in the following way:

$$ \begin{align*}
\mathbf{V}^T &= (\mathbf{V}_1^T, \ldots, \mathbf{V}^T_{n-1}) \\
\mathbf{r}^T &= (\mathbf{e}_{1,1}, \ldots, \mathbf{e}_{1,n-1})
\end{align*} $$

### 3.4 Subspace Iteration Method for Updating of Eigensolution

The a priori model and consequently also the modal coordinates of this model are updated in each iteration cycle of the approach. To reduce the numerical effort for this updating a subspace iteration method is applied (see e.g. Bathe 1986), which takes full advantage of some properties of the MLSQ-approach:

- The number of necessary eigensolutions is limited.
- A close approximation of the eigensolution is available from the previous iteration.
- The a priori model is changed only slightly after the first iteration steps, i.e. the eigensolution undergoes minor changes.

To solve eq. (7) the subspace iteration method starts from the form of the eigenvalue problem of eq. (1)

$$ \mathbf{M}' \mathbf{\Phi} \mathbf{u} \lambda^2 + \mathbf{C}' \mathbf{\Phi} \mathbf{u} \lambda + \mathbf{K}' \mathbf{\Phi} \mathbf{u} = 0 $$

The components of the modal matrix $\mathbf{\Phi}$ are divided into two parts which are corresponding to the displacements and velocities in the definition of the state vector in eq. (6)

$$ \mathbf{\Phi}_{u,i} = \lambda_i \mathbf{\Phi}_{u,i} $$

The components of the subspace iteration method are a simultaneous vector iteration of a subset of selected eigenvectors and their orthogonalization. The
basic equation for the simultaneous vector iteration is established in the following way

$$K'(k+1) = -M'(k)A(k) - C'(k)A(k),$$

(27)

where \( k \) denotes the \( k \)-th iteration step. Given a set of initial eigenvectors \( \Phi_u(k) \) and eigenvalues \( A(k) \), a new set of eigenvectors \( \Phi_u(k+1) \) can be obtained by solving this equation. Then, from these eigenvectors \( \Phi_u(k+1) \) an orthogonal set \( \Phi_u(k+1) \) and the eigenvalues are calculated. For this purpose a relation between these vectors of the form

$$\Phi_u(k+1) = \Phi_u(k+1)E$$

(28)
is assumed. The matrix \( E \) is in general complex valued. The assumption leads to the eigenvalue problem

$$\tilde{M} \tilde{E} A^2 + \tilde{C} \tilde{E} A + \tilde{K} E = 0$$

(29)

where the system matrices are defined by

$$\tilde{M} = \Phi_u^T M' \Phi_u \quad \tilde{C} = \Phi_u^T C' \Phi_u$$

$$\tilde{K} = \Phi_u^T K' \Phi_u.$$  

(30)

(31)

The Hermitian system matrices have the dimensions \([q \times q]\), where \( q \) is the number of eigenvectors of the subspace. For numerical reasons the number \( q \) is larger than the number of desired eigensolutions \( m \) (see e.g. Bathe 1986). The eigenvalues \( A \) are also eigenvalues of the original system. These eigenvalues and the matrix \( E \) are calculated by solving the eigenvalue problem

$$\begin{pmatrix} E_i \\ E_i \lambda_i \end{pmatrix} \lambda_i = \begin{pmatrix} 0 & I \\ -\tilde{M}^{-1} \tilde{K} - \tilde{M}^{-1} \tilde{C} \end{pmatrix} \begin{pmatrix} E_i \\ E_i \lambda_i \end{pmatrix},$$

(32)

which can be solved by standard approaches for small systems. The number of iterations of the subspace method depends significantly on the quality of the eigensolution of the previous step, i.e. starting information reduces the number of subspace iterations. The factorization of the stiffness matrix in eq. (27) is an important factor for the numerical effort of the scheme. Since especially the stiffness properties of the a priori model are not changed drastically after some iterations of the MLSQ-approach this factorization must not be updated in each step.

4. NUMERICAL EXAMPLE

4.1 Investigated System

The investigated system is a FE-model of a crane structure (see Fig.1), which consists of 828 three-dimensional beam elements with 1692 active DOF. The damping characteristics of the crane are significantly influenced by the cables (see Fig.1). Therefore in the FE-model three local Voigt-Kelvin dampers are added in parallel to the stiffness model of the cables 1-3. These dampers and some components of the stiffness matrix of the structure are identified based on simulated measurements. The global matrices of the three dampers \( C_{1,1}, C_{1,2} \text{ and } C_{1,3} \) are combined and identified with one factor \( c_1 \). The corrected parts of the stiffness matrix are (see Fig.1) the three cables (submatrix: \( K_1 \)), the superstructure (\( K_2 \)), the tower (\( K_3 \)), the jib (\( K_4 \)) and eight elements at the base (\( K_5 \)). The excitation force was located at the end of the jib (see Fig.1, node 1076) and applied in \( x-, y- \text{ and } z \)-direction. It is combined of 13 sinoidals, which are located in the neighborhood of the first eight eigenfrequencies of the system (frequency range: 0.1-3.0 Hz). The first eight eigensolutions are selected for all iteration steps. The simulated response was observed at eight nodes of the structure in \( x-, y- \text{ and } z \)-direction (nodes: 68, 132, 204, 1020, 1059, 1076, 1078 and 1082).

4.2 Results and Discussion

In both estimation steps of the MLSQ-approach 200 time steps with a time step length of \( \Delta t = 5 \cdot 10^{-2} \text{s} \) are assumed to be measured. The robust qualities of the approach especially in case of poor initial knowledge are shown, in premultiplying the exact submatrices by 0.2. This means that the exact correcting factors are equal to 5. The initial values are set to 1. The convergence behaviour of the correcting factors is shown in Fig.2 and the values after 30 and 200 iterations are listed in Tab.1. The main part of the correction is done already after 30 iterations, i.e. the relative errors of the correcting factors are smaller than 10 %. The relative errors after 200 iteration steps are nearly zero. The calculation time for one iteration step is in this example appr. 50 \text{s} (Workstation HP/700).
5. CONCLUDING REMARKS

(I) The proposed MLSQ-approach makes it possible to identify midsized FE-models based on time domain measurements by avoiding the severe numerical problems like computational speed and divergence, which are introduced by the model size. The method was shown to be robust and rather insensitive with respect to poor initial parameters.

(II) Since nonclassically damped systems are assumed throughout the method, a general viscous damping matrix is allowed and local dampers can be identified.

(III) Since the proposed algorithm is formulated in a restricted set of modal coordinates in each iteration step the complex eigenvalue problem must be updated. This means that the main numerical effort is shifted to this step and the efficient solution is quite essential. The subspace iteration method was successfully applied to solve this problem. The extension of the method to nonclassically damped systems with a general damping matrix is not complicated and the additional numerical effort is low.

(IV) A nonclassically damped FE-model of a crane structure with 1692 DOF was successfully identified. The algorithm calculated five stiffness and one damping parameter of the FE-model even if the initial knowledge about the parameter values is poor.

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7. APPENDIX

The Hermitian system matrix $\mathbf{U}$ in eq. (15) reads

$$\begin{pmatrix}
G_0 & W & 0 & \cdots & 0 & 0 & 0 \\
W^* G_1 & W & 0 & \cdots & 0 & 0 & 0 \\
0 & W^* G_1 & W & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & W^* G_1 & W \\
0 & 0 & 0 & \cdots & 0 & W^* G_2 \\
\end{pmatrix}$$

(33)

where $W_{i,j} = -e^{-\gamma_j \Delta t_i} \delta_{i,j}$

$$G_0 = I + B^* T B$$

$$G_1 = I + B^* T B + W^* W$$

$$G_2 = B^* T B + W^* W$$

$$B = \Psi^T T^T T^\Phi$$

(34)

The symbol $\delta$ denotes the Kronecker-Delta. The right hand side $\mathbf{u}$ in eq. (15) is evaluated as:

$$\mathbf{u} = \begin{pmatrix}
B^* T_1 (t_1) + r_2 (t_1) \\
B^* T_1 (t_2) + r_2 (t_2) t W^* T r_2 (t_1) \\
B^* T_1 (t_{n-1}) + r_2 (t_{n-1}) t W^* T r_2 (t_{n-2}) \\
B^* T_1 (t_{n-1}) t W^* T r_2 (t_{n-1}) \\
\end{pmatrix}$$

(35)

where

$$r_1 (t_i) = \psi^T T^T y (t_i)$$

$$r_{2,j} (t_i) = \int_{t_i}^{t_{i+1}} e^{\gamma_j (t_{i+1} - \tau)} \sum_{k=1}^{n_\text{at}} \Phi_{k,j} c_k (\tau) d\tau$$

(36)

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*Real-Time System Identification of Degrading Structures*  

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Table 1: Correcting factors from simulation ($s$) and identified values of the MLSQ-approach ($id$) after 30 and 200 iteration steps.

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Figure 1: Finite element model of crane structure

Figure 2: Factors $c_1(u)$ and $k_i(1)$