A LOCAL RESTORING FORCE SURFACE METHOD

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Abstract
The Restoring Force method and equivalent Force-State Mapping Technique were proposed to identify nonlinear mechanical systems [1-5]. The final result of the methods was a modeling of the nonlinear force in function of the displacement and the velocity. The first method proposed a polynomial series while the second expressed the model by its values on a grid. In this paper a generalized and combined approach of both methods is presented. The new method identifies the nonlinear force using a local nonparametric representation, while the mass is identified globally from the same data. This results in a set of quasi uncoupled equations which reduces the calculation effort significantly. The uncertainty due to stochastic errors (noise) and systematic model errors are studied. The benefit of a uniform phase plane covering is demonstrated and by means of simulations, it is shown how close a well chosen multisine displacement [6] approaches these properties.

Nomenclature

Overdots denote differentiation with respect to time. Caret brackets \( <,> \) denote mean values and circumflexes denote estimators. A superscript accent stands for the transpose operation. Cell indices indicate a belonging to a grid element. Capital letters indicate matrices and vectors. The Greek capital operator \( \Lambda \) expresses the systematic error of an estimator

- \( m \): dynamic mass
- \( x \): displacement
- \( \dot{x} \): acceleration
- \( f \): excitation force
- \( g \): nonlinear force, restoring force
- \( n_1 \): number of grid subdivisions in the x-dimension
- \( n_2 \): number of grid subdivisions in the x-dimension
- \( \delta x \): size of a grid element in the x-dimension
- \( u \): length of the grid in the x-dimension
- \( v \): length of the grid in the x-dimension
- \( A \): acceleration data vector
- \( F \): excitation force data vector
- \( G \): restoring force data vector
- \( S \): observation matrix
- \( P \): parameter vector

- \( E \): error vector
- \( M \): Gaussian white noise vector
- \( I \): unity matrix
- \( N \): number of simulations/experiments
- \( \sigma_f \): standard deviation of the force measurement noise
- \( \gamma_{pq} \): grid element numbered p and q
- \( \text{var} \): variance
- \( \text{rvar} \): relative variance

1. Introduction

Most mechanical systems are approximated by a linear model for modal analysis purposes. However, when these systems are subject to large displacement amplitudes, non-linear effects become important and the linear model consequently fails. Even when the amplitudes remain restricted, some non-linear distortions may occur, caused by the phenomenon of dry friction since ideal dry friction is independent of kinematic amplitudes but opposite to the sign of velocity.

Besides this rather undesirable non-linear behaviour, some industrial mechanical systems are made intentionally nonlinear. Nowadays the industrial shock absorbers are a remarkable example of this concept. The helical orifice, through which the oil is pushed during impact, provides a progressive damping behaviour in contrast with conventional damping, the damping force remains approximately constant during impact. This has the advantage of reducing the brake time considerably. Another example under development are the nonlinear vibration absorbers. Compared to linear versions, they may engender an enlargement of the suppression bandwidth [7].

In order to perform a valid modal analysis for such systems, an accurate and detailed non-linear model is needed. An accurate and detailed measurement of the non-linear characteristics of the system will contribute to the design of these systems with a deeper understanding and in a later stage in the proper design of a sound control strategy. The means of a detailed measurement is not confined to pure detection of non-linearities, e.g. Harmonic Detection Method [8] and the Corehence [9], but if possible to its unambiguous modelisation.

Several techniques are proposed in the literature, among which the Restoring Force method, an interesting time domain method introduced by Masri and Caughey in 1979 [1]. This method is able to characterize non-linear forces dependent “f velocity as well as displacement. At the end of this procedure the nonlinear forces are fitted to a series of standard functions. Initially a series of two-
dimensional orthogonal polynomials, viz. Chebyshev polynomials, was proposed since they had the advantage that their associated coefficients were independent of the selected order. Later on, the use of ordinary polynomials was preferred since the calculations could then be speeded up with a factor of almost 100 \[2]\]

The Force-State Mapping Technique is an equivalent development method, developed in order to study nonlinear space structures [3-4]. The final representation of the nonlinear forces is no longer presented under the form of a functional series. But the state space is partitioned into a grid and the non-linear force is calculated as the average of the forces from all measurements within the considered element. Both approaches will be combined and generalized in this article and the intrinsic advantages will be discussed. Warden and Tomlinson used both approaches for distinct purposes: modelling and plotting [5].

2. Theory

2.1. Basic Philosophy of the Restoring Force Method

The Restoring Force method is governed by the law of conservation of impulse momentum

\[ m \ddot{x} = g(x(t), \dot{x}(t)) + f(t) \]  

where \( m \) is the dynamic mass, \( \ddot{x}, \dot{x}, x \) are the kinematic signals, i.e. acceleration, velocity and displacement, \( f \) is the applied excitation force. Of them are explicit functions of time, in contrary with the nonlinear force, \( g(x, \dot{x}) \), called restoring force. The main idea of the procedure is to identify the restoring force as a function of velocity \( \dot{x} \) and displacement \( x \) using measured force and kinematic data at discrete time instants and an estimated value mass.

When an accelerometer or an impedance head is used only acceleration is measured and the velocity and displacement have to be calculated by integration. Note that new laser techniques make it conceivable to measure displacement and/or velocity directly without mechanical interference with the system. In any case, a triplet \((\dot{x}_k, x_k, \ddot{x}_k)\) is obtained for every sampling instant \( t_k \) and these triplets are then processed in order to obtain the desired model.

2.2. Division of the phase plane into a regular grid

In accordance with the Force-State Mapping approach the phase plane \((x, \dot{x})\) is divided into a rectangular grid. The limits of the grid are determined by the phase plane trajectory. Assuming that both kinematic signals are not subject to any offset the grid is centred around the origin according to following equations for grid cell \((p,q)\) including all samples \((\dot{x}_k, x_k, \ddot{x}_k)\) with:

\[
\begin{align*}
\dot{x}_p - \frac{\delta x}{2} \leq & \dot{x} < \dot{x}_p + \frac{\delta x}{2} \\
x_q - \frac{\delta x}{2} \leq & x < x_q + \frac{\delta x}{2}
\end{align*}
\]

\((\dot{x}_p, x_q)\) are the \(w\)-ordinates of the centre of grid element \((p,q)\). \begin{align*}
\dot{x}_p = & -u + \delta x (p - 0.5) \\
x_q = & -v + \delta x (q - 0.5)
\end{align*}

Each grid cell has the same dimensions \((\delta x, \delta x)\)

\[
\begin{align*}
\delta x &= \frac{2u}{n_1} \\
\delta x &= \frac{2v}{n_2}
\end{align*}
\]

In order to centre the entire grid symmetrically around the origin a careful choice for the lengths \( u \) and \( v \) is to be made. They are chosen as the maximum of the absolute values of the time series of respectively displacement and velocity data \( n_1 \) and \( n_2 \) are the number of subdivisions of the grid in the \( x \)-direction and the \( \dot{x} \)-direction, respectively.

In order to produce local notations, the time series will be reorganised topologically. Cell indices will indicate which grid element is involved. For example, a triplet \((\dot{x}_p, x_q, \ddot{x}_k)\) is then rewritten as \((x_p^q, x_q^p, \ddot{x}_k^p)\), indicating that this sample triplet is the \(i\)-th triplet located within grid element \((p,q)\). The number of points within grid element \((p,q)\) is denoted with \(n_{pq}\).

2.3. Formulation of the local restoring force model

For convenience, vectorial notation is used. If we consider that \(A^p_q\) and \(F^p_q\) are the vectors of measured acceleration and excitation force data, \(S^p_q\) the observation matrix, \(P^p_q\) the parameter vector and \(G^p_q\) the restoring force vector, then equation (1) can be written in vectorial form, for \(p=1...n_1, q=1...n_2\), viz.

\[
\begin{align*}
A^p_q &= S^p_q P^p_q \\
F^p_q &= G^p_q P^p_q
\end{align*}
\]

Explicitly:

\[
\begin{align*}
A^q_p &= x_q^p \quad F^q_p = x_q^p \\
\ddot{x}_k^p &= \ddot{x}_k^p
\end{align*}
\]

The complexity of the procedure is dominated by the complexity of the observation matrix and the associated parameter vector. The initial approach of Crawley et al [3-4] calculated one average value from all samples within each grid element. This corresponds physically to the fitting of a local horizontal plane to the local data. A disadvantage of this method is that a pure linear model cannot be modelled without under and over specification. Therefore a more complex local observation matrix and parameter vector approach is proposed. While for the initial order model the parameter vector is reduced to a scalar and the observation matrix to a column vector with \(n_{pq}\) ones, these become now for the proposed first order model:

\[
S_{pq} = \begin{bmatrix}
1 & x_q^p & x_q^p \\
1 & x_q^p & x_q^p \\
... & ... & ... \\
1 & x_q^p & x_q^p \\
1 & x_q^p & x_q^p \\
1 & x_q^p & x_q^p \\
\end{bmatrix}, \quad P_{pq} = \begin{bmatrix}
a_{pq} \\
b_{pq} \\
c_{pq}
\end{bmatrix}
\]

Physically, the parameters \(b_{pq}\) and \(c_{pq}\) correspond to the local gradients of the restoring force surface. Theoretically, the differentiation procedure could go even further and include more
terms besides these linear terms. But the number of parameters will increase approximately with the square of the applied order. If all cross terms are included for a $n$-th order model, the number of parameters per element amount to \((n+1)(n+2)/2\). E. g. for a second, third and fourth order model the number of parameters come to 6, 10 and 15. Consequently, the total number of grid elements should decrease in order to obtain a sufficient number of samples within each grid cell. An obvious drawback of this is the loss of resolution in favour of the augmented complexity of the local model.

2.4. Estimation of the local parameters

At start the dynamic mass is assumed to be unknown and will be determined within the identification procedure. The identification is based on a weighted LSE (Least Square Estimator). Using the LSE an the pole linear equations of (5-6), a cost function \(K\), which calculates the squared difference between measured (index \(m\)) and modelled restoring force (no index), is formed:

\[
K = \sum \sum B_p^q B_m^p \text{ with } E_p^q = (m A_p^q - S_p^q p_p^q - r_p^q) \tag{9}
\]

Note that the kinematic signals are initially assumed to be exact, so no index is used for the appropriate vector and matrix. The motivation of this assumption will be given later (in paragraph 2.6). The supra accent stands for the transpose operation and the horizontal bar indicates the estimators.

Minimising this cost function leads to following solution:

\[
\begin{align*}
\hat{p}_p^q &= (S_p^q S_p^q)^{-1} S_p^q C_m^p \\
G_p^q &= \hat{m} A_p^q - F_m^p \\
\hat{m} &= \left( \sum \sum A_p^q B_m^p A_p^q \right)^{-1} \left( \sum \sum A_p^q B_m^p F_p^q \right) \\
B_m^p &= \left( I_p^q - S_p^q (S_p^q S_p^q)^{-1} S_p^q r_p^q \right)\tag{10}
\end{align*}
\]

\(I_p^q\) is a unity matrix with dimensions \((n_p \times n_p)\). It can be found that \(B_m^p\) is an idempotent matrix.

From this set of equations, it is not clear which advantages come out of this approach in comparison with former approaches. Therefore new global vectors and one global matrix are introduced, viz.

\[
F^{11} \begin{bmatrix} \begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{array} \end{bmatrix} \begin{bmatrix} A_{11}^{1} \\ A_{12}^{1} \\ \vdots \\ A_{1n_p^1}^{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
F^{12} \begin{bmatrix} \begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{array} \end{bmatrix} \begin{bmatrix} A_{21}^{1} \\ A_{22}^{1} \\ \vdots \\ A_{2n_p^1}^{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
F^{1n_p} \begin{bmatrix} \begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{array} \end{bmatrix} \begin{bmatrix} A_{n_p^1}^{1} \\ A_{n_p^2}^{1} \\ \vdots \\ A_{n_p^1 n_p^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

With these notations it is possible to write the estimate for the dynamic mass in a ‘lighter’ but more involved form:

\[
\hat{m} = (A'BA)^{-1} (A'B_m^p) \tag{12}
\]

If matrix \(B\) is closely observed, it is clear that the sparsity of this matrix depends on the resolution of the employed grid. If a large number of subdivisions is used, the matrix shall aspire to the diagonal form. In this case the solution of the estimation procedure is obtained from a set of uncoupled equations. If the number of grid elements decreases and the number of parameters to be estimated ( \(\leq\) length of \(A\) ) remain constant the calculation effort will increase. In the extreme case where we would choose for only one grid element, the calculation effort will be maximal. In fact, this is just the case of the classical Restoring Force Method (1-21).

2.5. Stochastic and systematic model errors

The estimated values will certainly not be exact, due to systematic errors and stochastic errors (i.e. noise). The true local model (index \(i\)) will probably need more higher order terms than presented in the observation matrix. These supplementary higher order terms will be exclusively included in matrices with index \(s\):

\[
S_{m}^{p_q^q p_p^q} = S_{s}^{p_q^q p_p^q} + S_{m}^{p_q^q p_p^q} + M_{m}^{p_q^q} \tag{13}
\]

with \(S_s^{p_q^q p_p^q} = \begin{bmatrix} (x_1^{p_q^q})^2 & (x_1^{p_q^q})^2 & \ldots & (x_1^{p_q^q})^2 \\
(x_2^{p_q^q})^2 & (x_2^{p_q^q})^2 & \ldots & (x_2^{p_q^q})^2 \\
\vdots & \vdots & \ddots & \vdots \\
(x_n^{p_q^q})^2 & (x_n^{p_q^q})^2 & \ldots & (x_n^{p_q^q})^2 
\end{bmatrix}
\]

\[
M_m^{p_q^q} = \begin{bmatrix} \mu^{p_q^q} p_p^q M_{1}^{p_q^q} \\
\mu^{p_q^q} p_p^q M_{2}^{p_q^q} \\
\vdots \\
\mu^{p_q^q} p_p^q M_{n}^{p_q^q} 
\end{bmatrix} \tag{14}
\]

With these new matrices and vectors, it is possible to express the systematic model error by calculating the expectation value of the LSE. Assuming that the noise vector \(M_{m}^{p_q^q}\) is Gaussian white noise with variance \(\sigma_f^2\), the systematic errors \(\Delta\hat{m}\) on the dynamic mass and \(\Delta p_q^q\) on the local parameter vector are then easily calculated as:

\[
\Delta\hat{m} = \left( \sum \sum A_p^q B_m^p A_p^q \right)^{-1} \left( \sum \sum A_p^q B_m^p S_s^{p_q^q p_p^q} \right) \tag{15}
\]

\[
\Delta p_q^q = (S^{p_q^q} S^{p_q^q})^{-1} S^{p_q^q} (\Delta\hat{m} A_q^q - S_s^{p_q^q p_p^q}) \tag{16}
\]

With a similar calculation, the variance of the estimations, due to stochastic disturbances, are obtained as the expectation value of the squared difference between the estimation and its expected value, producing

\[
\text{var}(\hat{m}) = \sigma_f^2 (A'BA)^{-1} \tag{17}
\]

\[
\text{var}(\hat{p}_q^q) = (S^{p_q^q} S^{p_q^q})^{-1} S^{p_q^q} (S^{p_q^q} S^{p_q^q})^{-1} \tag{18}
\]

If the mass is not estimated, equations 16 and 18 reduce into the well known LSE format:

\[
\Delta p_q^q = -(S^{p_q^q} S^{p_q^q})^{-1} S^{p_q^q} \hat{m} A_q^q \tag{19}
\]

\[
\text{var}(\hat{p}_q^q) = \sigma_f^2 (S^{p_q^q} S^{p_q^q})^{-1} \tag{20}
\]

2.6. Motivation for a uniform phase plane covering

In [6] a class of excitation signals is presented that causes multisine
displacement through the implementation of a software feedback.

Some optimisation methods are presented for the associated phase spectrum as to uniformise the phase plane covering. This is necessary for the success of the proposed Local Restoring Force model in order to dispose over a sufficient number of samples for each grid element. The advantages of a multisine displacement are abundant in comparison with classical excitation signals:

- full control of the power spectrum of the excitation/displacement
- simple integration/differentiation of the measured signals
- optimal covering of the phase plane through the particular choice of the phases
- improvement of the signal-to-noise ratio by putting the non-excitation lines to zero

Because the bandpass spectrum multisine will consist of only a few excitation frequencies, this will allow the user of measuring the displacement very accurately and strengthens the assumption of an (almost) exact known displacement (see section 2.4).

Furthermore a uniform covering induces some supplementary features in the case without mass estimation. In this context, the matrices \((S^pS'^p)^{-1}\), that play an important role in the error magnitude of the estimations, are studied. A relevant parameter for the local model is the constant \(a^0\), its expected value and variance. The variance is calculated from (20) resulting in (21) for a first order model.

\[
\text{var}(a^0) = \frac{\alpha \cdot \sigma^2}{\alpha \cdot n_{pq} - \beta}
\]

\[
\alpha = \sum_i (x_i^p)^2 \sum_i (x_i^p)^2 - \sum_i (x_i^p x_i^q)
\]

\[
\beta = \sum_i x_i^p x_i^q \sum_i x_i^p x_i^q - \left( \sum_i x_i^p \right)^2 \sum_i x_i^q - \left( \sum_i x_i^q \right)^2 \sum_i x_i^p
\]

It can be proven that the minimum of the variance is obtained if \(\beta\) is zero (see appendix). To make \(\beta\) vanish, the sum of the displacement co-ordinates and the sum of the velocity w-ordinates have both to be zero (24). The related minimum is equal to the variance of the zero order model.

\[
\sum_i x_i^p = \sum_i x_i^q = 0 \implies (\beta = 0) \implies \text{var}(a^0)_{\text{min}} = \frac{\sigma^2}{n_{pq}}
\]

In order to fulfil both conditions, the local gravity centre co-ordinates should be deducted from the original local co-ordinates (equation 25). In this way the mean or offset is removed from the local co-ordinates and the procedure is caused to calculate the restoring force in the local gravity centre. Similarly, an important improvement can be obtained for the systematic error on the parameters for a zero order model. In that case the systematic error in the zero order model will equal the systematic error of the first order model.

\[
S_{\text{local}}^p = S^p - \sum_{pq} x_i^p \left[ \begin{array}{ccc} 0 & 0 & n_{pq} \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccc} x_i^p \\ x_i^q \end{array} \right]
\]

It can be stated that a uniform phase plane covering is important for two reasons. Firstly the local parameter model needs a minimum number of sample triplets in each grid element. Notice that if higher order models are used, the minimum number increases with the square. In second instance, a judicious choice for the distribution of the samples enables a minimum variance of the estimates for the first order model. In complement a uniform covering reduces the magnitude of the zero order model error to the magnitude of the first order model error. However the distribution of sample triplets within the grid cell is totally irrelevant for the zero order variance and first order model error.

The condition sine qua non of these statements is exclusion of mass estimation. Otherwise additional terms, including the vector of acceleration \(\dot{A}\), appear in the concerned expressions (17-18). A uniform covering will no longer enable minimum variance and model error.

2.7. Simulation set-up

In order to verify the above mentioned features of the local restoring force method, some simulations are carried out. A uniform phase plane covering is obtained by sampling 2048 times over one period a phase plane trajectory, which was derived from a particular multisine displacement (Fig.1). The multisine is calculated from a flat band pass velocity spectrum. with five components distributed regularly from 20.5Hz to 22.0Hz with a resolution of 375mHz. The period of the signal is obtained from the largest common divider for the frequencies and results in 8 seconds. Note that a minimum number of frequency components is advisable to reduce the amount of intermodular distortion. The uniformity of the covering is optimised by using the Differentiated Band Optimisation technique during 100 iterations [6].

![Fig.1 : Sampled phase state trajectory](image)

The local restoring force model is constructed on an arbitrary, symmetrical 10 by 10 grid, formed as section 2.2 explains, thus generates 100 grid elements. Because of the elliptical
circumscribing curve of the phase plane trajectory not all elements contain samples. Fig. 2 gives the distribution of samples within the grid elements. It is observed that 12 elements do not contain any sample point and will therefore not participate to the identification procedure. Besides these empty cells, the other active cells contain 2048/88 = 23 samples each on the average. Nevertheless the central gap and the circumscribing curve reduces this amount for the elements, that are involved with these irregularities. Besides this, it is observed that the other ‘normal’ cells do not have an equal number of samples. It varies from 17 to 32 points per cell.

Fig. 2: Distribution of number of samples per cell (10 x 10 grid mapping of Fig. 1)

Because a zero and first order model are proposed, a second and third order nonlinearity are used to study the influence of systematic model errors:

- **Quadratic nonlinearity**: \( g(x) = a(x - bx^2) \)  
- **Cubic nonlinearity**: \( g(x) = a(x - cx^3) \)

The exact function values, calculated over the employed grid, are shown in Fig. 3 and 4. Both nonlinearities are used to accentuate the difference between odd and even nonlinearities. Since, it can be predicted that e.g. a fourth order nonlinearity exhibits a similar behaviour as the quadratic nonlinearity. The same can be said about higher order terms or cross terms. A more or less arbitrary choice with values of \( 20, 1e3 \) and \( 1e6 \) is made for the constants \( a, b \) and \( c \).

When a dynamic mass of \( 1e3 \) kg is taken, the time series of the restoring force, the inertia force and the excitation force relate in peak-to-peak values as \( 3, 7 \) and \( 7 \).

Fig. 3: Quadratic restoring force

During the simulations, the kinematic signals (i.e. acceleration, velocity and displacement) are considered to be exact. Gaussian white noise with a standard deviation of 1% of the peak value of the excitation force, is added to the time series of the excitation force. Over a series of 100 simulations, the (sample) mean and (sample) variance of the restoring force estimates are recorded for each case. The systematic error is calculated as the difference between mean and exact values (equations 28, 30). \( \Delta (p,q) \) is one realisation of the estimation of a local restoring force value in the grid element (p,q).

\[
\text{Mean: } \langle \hat{a}_{pq} \rangle = \frac{1}{100} \sum_{i=1}^{100} \hat{a}_{pq,i} \tag{28}
\]

\[
\text{Model error and Bias: } \Delta \hat{a}_{pq} = \langle \hat{a}_{pq} \rangle - a_{pq} \tag{29}
\]

\[
\text{Relative variance: } \text{rvar} (\hat{a}_{pq}) = \frac{\sum_{i=1}^{100} (\hat{a}_{pq,i} - \langle \hat{a}_{pq} \rangle)^2}{100 (\sigma^2_{pq}/n_{pq})} \tag{30}
\]

For each case, the distribution of the properties, systematic error and (relative) variance, are shown in Fig. 5 for the quadratic nonlinearity (index q) and in Fig. 6 for the cubic nonlinearity (index c).

**2.8. Discussion of the simulation results**

For sake of simplicity, the abbreviations or index series will be used throughout the discussion. E.g. when we focus attention on the bias in regular w-ordinates in the case of the cubic nonlinearity with no mass estimation, we will speak about the bias of \( cm0 \) and \( cm1 \) (see also Fig. 6). When these distributions are focussed, it is clear that a first order model results in a much lower bias than a zero order model (error of up to \( 1.8 \) against \( 1.4 \)). These conclusions are fortified when additionally the mass is estimated in parallel with the restoring force \( cm0 \) and \( cm1 \). But this is because the
magnitude of the bias on the mass estimation is important (table 1: 2.6e-4kg). More drastically, in the quadratic case the mass estimation yields a totally wrong value, even the order of magnitude differs!

A second difference between first and zero order estimations speaks in favour of the latter. The relative variance is minimal for the zero order. In fact, if an infinite number of experiments would be performed, this relative variance would be exactly 1 for all grid elements. For a finite number of simulations N, the variance of the relative sample variance is easily calculated as:

$$\text{var}(\text{rvar} (a^p)) = \frac{2}{N}$$

(31)

This means that due to the stochastic disturbances, the exact value cannot be measured but the results will be spread out over some interval. For N sufficiently large, the relative variance is approximately normally distributed and the 95% confidence interval for the variance is approximately given by 2 times the standard deviation, i.e.

$$95\% \text{ confidence interval : } [1 - 2 \left( \frac{2}{N} \right)^{\frac{1}{2}}, 1 + 2 \left( \frac{2}{N} \right)^{\frac{1}{2}}]$$

(32)

For N=100, the 95% confidence interval turns out to be [0.72,1.28] and this is governed by the variance of the estimates of the parameters. Although most first order estimates have a variance near one, some cells exhibit a very high variance up to ten times the zero order minimum (see variances qnl, cmg1).

The attractive feature about estimating the parameters in the gravity co-ordinates, is that both properties become equal and minimal for both zero and first order. It doesn’t make any difference for the quality of the estimates whether a zero or a first order local model is used when estimating in gravity w-co-ordinates (compare qng0,qng1 and cmg0-cmg1). In this way, we dispose over the most advantageous properties of both models at the same time. The reader can check that this is not true for higher order models!

Table 1: Mass estimation : mean and standard deviation

<table>
<thead>
<tr>
<th>mass estimation (kg)</th>
<th>zero order</th>
<th>first order</th>
</tr>
</thead>
<tbody>
<tr>
<td>quadratic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>9.02E-6</td>
<td>9.99E-4</td>
</tr>
<tr>
<td>standard deviation</td>
<td>1.9837</td>
<td>4.78E-7</td>
</tr>
<tr>
<td>cubic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>7.39E-4</td>
<td>1.01E-3</td>
</tr>
<tr>
<td>standard deviation</td>
<td>3.66E-6</td>
<td>9.40E-6</td>
</tr>
</tbody>
</table>

Also, these advantages no longer exist, if the mass is estimated, since the acceleration data is now involved. A uniform phase plane covering has no explicit consequences for the acceleration data. It means that the local acceleration data will not be supplied in an optimal format to reduce model errors and variance. In these examples, where the inertia force is considerably large in comparison with the restoring force, the properties are affected in an important way by the estimates of the mass.

The estimates of a first order generates a much lower bias and model error than a zero order (cmg0-cmg1, qng0-qng1). No obvious differences are observed between the variances, recorded in regular co-ordinates and in gravity co-ordinates (cmg0-cmg0, cmrl-cmg1, qmr0-qmg0, qmr1-qmg1). Nevertheless, estimating the mass adds some ‘variance’ to the variance of the restoring force estimates (zero order rvar up to 4 against 1.5 and first order rvar up to 16 against 10). In additional simulations, it was shown that for very light structures (with a small inertia m), the estimates of the mass no longer influences the estimates of the nonlinear force and hence the previous conclusions are valid again.

Finally, if the results on the cubic and quadratic nonlinearities are compared, the offset of the bias distribution in the quadratic case is much more pronounced than in the cubic case, especially for the first order model (qnl1, qng1, qnl1, qng1). In other words, the bias is concentrated around a non-zero value. This is a direct consequence of the even character of the quadratic nonlinearity.

Table 2: Evaluation of model error

<table>
<thead>
<tr>
<th>model error</th>
<th>without mass estimate</th>
<th>with mass estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>co-ordinates</td>
<td>regular</td>
<td>gravity</td>
</tr>
<tr>
<td>zero order</td>
<td>poor</td>
<td>very good</td>
</tr>
<tr>
<td>first order</td>
<td>very good</td>
<td>very good</td>
</tr>
</tbody>
</table>

Table 3: Evaluation of stochastic error

<table>
<thead>
<tr>
<th>variance</th>
<th>without mass estimate</th>
<th>with mass estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>co-ordinates</td>
<td>regular</td>
<td>gravity</td>
</tr>
<tr>
<td>zero order</td>
<td>excellent</td>
<td>excellent</td>
</tr>
<tr>
<td>first order</td>
<td>excellent</td>
<td>excellent</td>
</tr>
</tbody>
</table>

In tables 2 and 3, a qualitative evaluation is given of the simulation results. These tables resume the figures 5 and 6. The total error will include both errors, so the reader should minimise the sum by selecting the appropriate model.

3. Conclusions

In this article, a new identification method for nonlinear, mechanical systems is presented. The Local Restoring Force model can be seen as a beneficial combination of the Force-State Mapping approach and Al-Hadid’s variant of the initial Restoring Force model of Masri et al. Al-Hadid’s functional series representation is used to model the nonlinear force within local topologies, defined by Crawley’s grid representation. In this way, an optimal combination can be provided, together with the possibility of estimating the dynamic mass globally from a set of quasi uncoupled equations. When a certain number of samples is recorded, several local models can be used. The experimenter will have to make a judicious choice between several model parameters:

* with or without estimation of the mass
* resolution of the applied grid, eventually differentiating the number of divisions in both dimensions (displacement and velocity)
* complexity of the local model (order 0, 1, . . .)
* using regular co-ordinates or gravity centre co-ordinates

This choice will influence the properties of the obtained results as shown. Excitation signals that provide uniform phase plane covering are very interesting because they allow certain additional features. It is possible to construct an optimal covering from multisine excitation.

The grid resolution and local model order interplay in such a way that it is possible to obtain comparable results with different resolutions in combination with well chosen model orders,
depending on the focussed item. If the experimenter wants to reduce the model error, he can choose as well for a higher local model order, as for a higher resolution. Generally, an improvement of the model error results in a worsening of the variance and vice versa.

The original Force-State Mapping approach of Crawley et al corresponds with the zero order model local without mass estimation (qmr0, qmr0). To improve the systematic error a first order local mode can be used, but the variance will deteriorate (qmr1, qmr1). The latter is an argument to avoid higher order local models. An optimal solution with minimum model error and variance is obtained by performing the estimations in the gravity centre coordinates (qm00, qm10, qm01, qm11). When the mass is estimated, the systematic error and variance of the restoring force estimate is strongly affected by the systematic error and variance of the mass estimate, especially for non-linear structures (qm00, qm10, qm01, qm11). When the same is done in gravity co-ordinates, the systematic error can improve to some extend (qm00, qm10, qm01, qm11).

Variance and model errors are important properties to evaluate estimators. By making a proper choice of the additional degrees of freedom for the Restoring Force Method (grid resolution, local model complexity, ...) an optimal model can be proposed. The simulations try to show the reader the influence of the several local model parameters and to give him a certain feeling as how to select the concerned parameters.

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References


Appendix: Minimal first order variance

In order to prove statement (24), the local co-ordinates are uncoupled in two parts:

\[
\begin{align*}
\begin{bmatrix}
\hat{p}_q^1 \\
\hat{p}_q^2 \\
\hat{p}_q^3
\end{bmatrix} &= \begin{bmatrix}
\hat{x}_q^0 \\
\hat{x}_q^1 \\
\hat{x}_q^2
\end{bmatrix} + q_{n,q}
\end{align*}
\]

with \( q_{n,q} \) the local gravity a-ordinates (34)

If the first order variance is calculated from equation (20), it follows that:

\[
\text{var} (\hat{p}_q^0) = \sigma^2 \sum \hat{x}_q^2 p_0 \sum \hat{x}_q^2 (x_p^0) \sum \hat{x}_q^2 (x_p^0)^2
\]

Substitution of equation (33) in equation (35) and focussing attention on the variance of the first parameter, we obtain:

\[
\text{var} (\hat{p}_q^0) = \frac{\sigma^2}{\bar{y}^0} \frac{\sum \hat{x}_q^2 (x_p^0)^2 \sum \hat{x}_q^2 (x_p^0)^2}{\sum \hat{x}_q^2 (x_p^0)^2} \] (36)

\[
\alpha = \left[ \frac{\sum \hat{x}_q^2 (x_p^0)^2 \sum \hat{x}_q^2 (x_p^0)^2}{\sum \hat{x}_q^2 (x_p^0)^2} \right] \frac{\sum \hat{x}_q^2 (x_p^0)^2}{\sum \hat{x}_q^2 (x_p^0)^2} \] (37)

\[
\beta = \frac{\sum \hat{x}_q^2 (x_p^0)^2 \sum \hat{x}_q^2 (x_p^0)^2}{\sum \hat{x}_q^2 (x_p^0)^2} \] (38)

Since the term \( \beta \) is composed of the sum of two quadratic terms and \( \alpha \) of a positive definite product, the minimum is in consequent reached, if the term \( \beta / \alpha \) is zero. This is the case if both mean values, \( x \) and \( i \), are zero.

q.e.d.
Fig. 5: Histogram of model error and relative variance of a quadratic nonlinearity 
(m/n: with or without mass estimate; r/g: regular or gravity co-ordinates; 
0/1: zero or first order model)

Fig. 6: Histogram of model error and relative variance of a cubic nonlinearity 
(m/n: with or without mass estimate; r/g: regular or gravity co-ordinates; 
0/1: zero or first order model)