Estimation of the Number of Uncorrelated Sources by means of Time Series Modelling

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Abstract

Sound and vibration source identification is a very important aspect in sound and vibration reduction, and can be characterised as a multiple input/multiple output problem where only multiple output measurements are available, and the input characteristics have to be derived from the analysis of the output. Source identification includes estimation of the number of uncorrelated sources, determination of the location, behaviour and contribution of each source. In this paper, two new methods dealing with the determination of the number of uncorrelated sources from output data are presented. The output measurements are first modelled by a time series model (ARMAV model or ARV model), and then the cross spectral matrix of the output and the variance matrix of the model residuals are estimated from the model. Eigenvalue decomposition of these two matrices yields information about number of the uncorrelated sources. The theoretical background of these two methods is presented, and the validity of these two methods is checked by simulations. The results show the advantages of these two methods over the Principal Component Analysis of the cross spectral matrix obtained by FFT.

NOMENCLATURE

\( l \) number of sources
\( m \) number of the reference signals
\( n \) number of data records
\( N \) number of data in each record
\( S_{xx}(\omega) \) cross spectral matrix
\( S_{xx}^{ARV}(\omega) \) cross spectral matrix obtained from ARV
\( U(\omega) \) left unitary matrix of SVD
\( V(\omega) \) right unitary matrix of SVD
\( z_i \) reference signal
\( \{x\} \) reference signal vector
\( \{X\} \) Fourier transform of \( \{x\} \)
\( \{X_i(\omega)\} \) Fourier transform of the \( i \)th record of \( \{x\} \)
\( [X(\omega)] \) matrix in equation 1
\( y \) the output signal
\( \alpha(\omega) \) singular value matrix of \( S_{xx}(\omega) \) or \( S_{xx}^{ARV}(\omega) \)
\( \phi \) autoregressive parameter
\( \theta \) moving averaging parameter
\( \sigma \) variance matrix of \( \{x\} \)
\( \Delta \) sampling interval
\( \Sigma(\omega) \) singular value matrix of \( [X(\omega)] \)
\( \Sigma_2 \) eigenvalue matrix of \( \{x\} \)
\( \omega \) circular frequency variable

1 Introduction

Sound and vibration source identification is a very important aspect in sound and vibration reduction. It includes estimation of the number of uncorrelated sources, and determination of the location, behavior and contribution of each source. In this context, coherence analysis is a widely used technique[1]. In this technique, the source identification problem is modelled as a multiple input/single output problem. The inputs are the source reference signals which
are the structural responses (accelerations, strains, forces) measured in the vicinities of the physical sources, and the output is the sound pressure or vibration level at the interested position. The causal relations between the inputs and output are analysed by the coherence function, from which the dominant reference signals and their contributions can be determined.

When the reference signals are mutually uncorrelated, ordinary coherence analysis is applicable. For mechanical structures, this assumption is often violated. An example from the automotive field will illustrate this: in the analysis of drive train noise, every connection force between body and drive train can be considered as a source (input force). It is difficult to measure the forces directly, therefore accelerations or strains in the vicinities of the connection points are used as source reference signals. It is clear that in the situation of a car, the response at a reference point is influenced by other sources. The input reference signals are therefore correlated and the ordinary coherence analysis cannot be applied.

Partial coherence analysis can cope with the situation where the reference signals are correlated. It first forms a new set of conditioned uncorrelated signals by eliminating the linear relationships among the reference signals with a step by step Gauss elimination procedure. But the eliminating sequence needs a priori knowledge about the sources, and plays an important role to the final analysis results. With highly correlated signals, some computation problems such as matrix singularity may arise in the analysis.

As an alternative to the partial coherence analysis, virtual coherence analysis was developed [2]. It calculates a new set of uncorrelated virtual signals, referred to as virtual signals, by linear transformation, and overcomes the influence of eliminating sequence in the partial coherence analysis. The concept of virtual coherence was introduced to describe the linear causality between the virtual signals and the output signal. This technique was further developed as Principal Component Analysis (PCA) method [3–4]. In order to get a new set of uncorrelated virtual signals, the PCA technique forms a linear transformation by eigenvalue decomposition of the cross spectral matrix of the reference signals. It can overcome the computation problems in the partial coherence analysis and can also get rid of the influence of eliminating sequence. Furthermore, PCA method can yield a straightforward estimation of the number of the uncorrelated phenomena in the reference signals, and form a minimum set of uncorrelated virtual signals. This number will equal the number of the uncorrelated sources under some conditions, which can provide a deep insight into the sources. Combined with other techniques like operating deflection shape or acoustic field shape analysis, the PCA method has found successful applications in source identification [3–4].

Usually, the cross spectral matrix used in PCA is formed by transforming the time domain observations of the reference signals by means of FFT which will introduce some errors. In these errors, the leakage error is found to be the most important one and prevents PCA from yielding correct analysis results under some situations [5]. In order to get reasonable analysis results, a large number of averages are needed in the estimation of the cross spectral matrix. In practice, it means a long stable running state of the structure under the analysis is needed. This condition limits its application for the nonstationary signal analysis, such as the run-up and run-down sound or vibration analysis of vehicles.

In this paper, the theory of the PCA method for the determination of the number of uncorrelated sources is reviewed. Next, an alternative method based on time series analysis is put forward. This method first modelled the reference signals by a time series model, such as AutoRegressive Vector model (ARV) or AutoRegressive Moving Average Vector model (ARMAV). The cross spectral matrix of the reference signals, with less bias, no leakage error and high resolution when compared with that obtained by FFT, can then be obtained from the model. The information about the number of the uncorrelated sources can be acquired from the eigenvalue decomposition of the cross spectral matrix obtained from the time series model. This method only needs one time record of the reference signals, and produces no leakage error. It can in principle be used for the analysis of nonstationary signals. The number of uncorrelated sources can also be determined by the eigenvalue decomposition of the variance matrix of the model residual vector in time domain. The theoretical background of the methods is presented, and checked by simulations. The results show the advantages of these methods over the FFT-based PCA method.

2 Theory of PCA for the determination of the number of uncorrelated sources

The model of source analysis problem is shown in Figure 1. The output y is the sound pressure or the vibration level at the target position, the inputs \( f_1, \cdots, f_l \) are the real sources which can not be measured directly and need to be analysed, and \( x_1, \cdots, x_m \) are the reference signals of sources.

Figure 1 shows that the source analysis problem can be roughly divided into two steps:

1. determination of reference signals

This is a multiple input/single output problem. Before the analysis, some sources can be determined according to a priori knowledge, but only a set of reference signals \( x_1, \cdots, x_m \) which are the responses of real sources through a system \([H] \), can be measured because the structure under analysis is complex, as well as the measurement environment. The multiple coherence function between the output signal and the given reference
signals in a practical situation should be sufficiently high, in order for the theoretical assumptions and later conclusions to be reasonable.

(2) Source analysis

This is a multiple input/multiple output problem, where the outputs are the reference signals and the inputs are the real sources. The aim is to obtain some information about the real sources from the analysis of the reference signals. This analysis is an operating data analysis problem, because only the multiple outputs are available. The estimation of the number of uncorrelated sources belongs to this aspect in the source analysis.

Suppose \( \{ x \} \) is a vector of the reference signals, the number of uncorrelated phenomena in \( \{ x \} \) can be obtained by forming a data matrix as

\[
[X(\omega)] = [(X_1(\omega)), \cdots, (X_n(\omega))]
\]  

(1)

in which \( \{ X_i(\omega) \} \), \( i = 1, \cdots, n \) is the Fourier transform of the \( i \)th record of reference signals measurements, \( n \) is the number of records. Using the Singular Value Decomposition (SVD), \( \{ X_i(\omega) \} \) can be decomposed as the following.

\[
[X(\omega)] = [U(\omega)]\Sigma(\omega)[V(\omega)]^H
\]  

(2)

where \( [U(\omega)] \) and \( [V(\omega)] \) are unitary matrices, and \( [\Sigma(\omega)] \) is a diagonal matrix which has the singular values of the matrix \( [X(\omega)] \) in descending order. The number of non-zero singular values is the number of uncorrelated phenomena in \( \{ x \} \).

The cross spectral matrix of the reference signals \( \{ x \} \) is

\[
[S_{xx}(\omega)] = \frac{1}{nN} [X(\omega)][X(\omega)]^H
\]  

(3)

where \( H \) represents conjugate transpose, and \( N \) is the length of each data record. The number of uncorrelated phenomena in \( \{ x \} \) can also be found by the eigenvalue decomposition of the cross spectral matrix \( [S_{xx}(\omega)] \).

\[
[S_{xx}(\omega)] = [U(\omega)][\Sigma(\omega)][U(\omega)]^H
\]  

(4)

in which \( [U(\omega)] \) is the eigenvector matrix, and \( [\Sigma(\omega)] \) is the eigenvalue matrix. The number of non zero eigenvalues is the rank of the matrix \( [S_{xx}(\omega)] \), as well as the number of uncorrelated phenomena in \( \{ x \} \).

In practice, no eigenvalue and singular value will be really zero, due to the influence of the noise in the measurements of the reference signals and the errors in the computation. Considering this, the SVD of the matrix \( [X(\omega)] \) or the eigenvalue decomposition of the matrix \( [S_{xx}(\omega)] \) is calculated at each discrete frequency, and the eigenvalues or the singular values are plotted as the functions of frequency. The number of uncorrelated phenomena in the reference signals is approximately determined by the number of the significant eigenvalues or the singular values in the plots.

Besides the number of uncorrelated phenomena in the reference signals, the number of uncorrelated sources can also be determined under some conditions. The relation between the real sources \( \{ F \} \) and the reference signals \( \{ X \} \) can be described in the frequency domain as following

\[
[X]_{m \times 1} = [H^*]_{m \times l} [F]_{l \times 1}
\]  

(5)

in which \( [H^*] \) is the frequency response function matrix, \( l \) and \( m \) are the numbers of the real sources and the reference signals respectively. If the column vectors of the matrix \( [H^*] : \{ H^*_1, \cdots, H^*_l \} \) are independent, and if

\[
m \geq l
\]  

(6)

namely the number of the reference signals is larger than the number of the uncorrelated sources, then

\[
\text{rank}([H^*]_{m \times l}) = l
\]  

It can be deduced from equation 5 and 6 that the number of uncorrelated phenomena in \( \{ F \} \) equals to that in the reference signals \( \{ X \} \). It means that the number of uncorrelated phenomena in \( \{ F \} \), or the number of uncorrelated sources, can also be found from the SVD of the matrix \( [X(\omega)] \) or the eigenvalue decomposition of \( [S_{xx}(\omega)] \). The condition in equation 6 is usually satisfied. Otherwise the reference signals has omitted the influence of some sources which can be checked out by the multiple coherence function between the output signal and the reference signals in the determination of reference signals.

It can be seen from equation 1 and 3 that in order to determined the number of uncorrelated phenomena in the reference signals, the number of data records employed to form the matrix \( [X(\omega)] \) and \( [S_{xx}(\omega)] \) should be larger than the number of the uncorrelated phenomena in the reference signals or the number of the uncorrelated sources, otherwise the FFT-based PCA will not yield useful information. In practice, in order to decrease the influence of the random errors in measurements, the needed number of data records is very large, usually as large as possible. Therefore, the system under analysis should be stable in time. For nonstationary problem, such as the sound and vibration analysis of run-up and run-down of vehicles, this method is not applicable.
Usually, the matrix $[X(\omega)]$ and the cross spectral matrix $[S_{xx}(\omega)]$ are acquired FFT. Under some assumptions, FFT will introduce some errors\cite{6}. These errors are inherent to FFT, and also limit the applications of the FFT-based PCA method.

3 Time series analysis method

Time series analysis method is an efficient technique to analyze operating data. It models the operating data as the output of a system excited by white noise inputs. The models used to describe the system are difference models\cite{7-8}. For the source identification problem where only outputs of the system are available, the time series analysis method is suitable. For convenience, from now on, the reference signals are also referred as the time series. When dealing with a single time series, autoregressive models (AR), or autoregressive moving average models (ARMA) can be used, otherwise Autoregressive vector (ARV) or autoregressive moving average vector (ARMAV) models should be used. When a statistically adequate model is determined, some information about the time series or the system generating the time series can be obtained from the model, such as power spectrum of the time series, modal parameters of the system generating the time series. Furthermore, the model can also be used for prediction of future values of the time series, optimal control of the system generating the time series. In this paper, the time series analysis method is used for the source identification.

The motivation of using the time series analysis method for source identification is that the time series analysis method has some advantages compared to the FFT method. It is well known that FFT will introduce some errors. But when using an adequate model in the time series analysis method, a less-biased and no leakage error result with higher resolution can be obtained. The most important factor is that the time series analysis method is most suitable for the analysis of time series with short duration, and can be used for the analysis of nonstationary time series. In this section, the theoretical background of the time series analysis method for source identification is given. Besides the eigenvalue decomposition of the cross spectral matrix of the reference signals as in the FFT-based PCA method, a new method based on the analysis of model residuals is also introduced and described in the next section.

For the multiple time series $\{x\}$ (a $m \times 1$ vector), an ARMAV model can be used\cite{7-9}.

$$\{x_t\} + [\phi_1]\{x_{t-1}\} + \cdots + [\phi_p]\{x_{t-p}\} = \{a_t\} + [\theta_1]\{a_{t-1}\} + \cdots + [\theta_q]\{a_{t-q}\}$$

(7)

in which, $p$ is the model order; $[\phi]$ is a $m \times m$ matrix, the autoregressive parameter of the model; $[\theta]$ is a $m \times m$ matrix, the moving average parameter of the model; $\{a_t\}$ is the model residual vector, a $m \times 1$ white noise vector function of time which satisfies the following equation.

$$E[\{a_i\}\{a_j\}^T] = \begin{cases} \sigma_a^2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(8)

where $\sigma_a^2$ is the variance matrix of $\{a_t\}$. An ARV model can also be used.

$$\{x_t\} + [\phi]\{x_{t-1}\} + \cdots + [\phi_q]\{x_{t-q}\} = \{a_t\}$$

(9)

in which $q$ is the order of the ARV model. Theoretically, these two models are equivalent but the order of the ARV model is much higher than that of the ARMAV model. In practice, the ARV model is preferred for the linear procedure of its parameter estimation. For this reason, we only use the ARV model in our analysis. Let

$$[\phi(B)] = (I + [\phi_1]B + \cdots + [\phi_q]B^q)^{-1}$$

(10)

in which, $B$ is the backward operator

$$\{x_t\}_B = \{x_{t-1}\}$$

then the ARV model can simply be written in a matrix notation,

$$[\phi(B)]\{x_t\} = \{a_t\}$$

(11)

From the model above, the cross spectral matrix of the time series $\{x\}$ can be deduced as\cite{8}:

$$[S_{xx}^{ARV}(\omega)] = \frac{\Delta}{2\pi}[\phi^{-1}(\omega)][\sigma_a^2][\phi^{-H}(\omega)]$$

(12)

in which, $H$ represents conjugate transpose; $\Delta$ is the sampling interval of the time series.

In analogy to the FFT-based PCA method, the number of uncorrelated phenomena in the time series can also be found from the the eigenvalue decomposition of its cross spectral matrix $[S_{xx}^{ARV}(\omega)]$:

$$[\sigma_{xx}^{ARV}(\omega)] = [U(\omega)][\alpha(\omega)][U(\omega)]^H$$

(13)

where $[U(\omega)]$ is the eigenvalue matrix, and $[\alpha(\omega)]$ is the eigenvalue matrix. Because the cross spectral matrix $[S_{xx}^{ARV}(\omega)]$ is obtained from the model, it is a less-biased estimation with high resolution and no leakage error\cite{7}. Eigenvalue decomposition of $[S_{xx}^{ARV}(\omega)]$ can yield more accurate information on the number of uncorrelated phenomena in the time series, as well as the number of the uncorrelated sources when the time series method is used for the source identification.

4 Time Domain Analysis

In equation 11, matrix $[\phi(\omega)]$ is a full rank matrix because its inverse matrix exists, therefore

$$\text{rank}([S_{xx}^{ARV}(\omega)]) = \text{rank}([\sigma_a^2])$$

(13)
Considering the eigenvalue decomposition of \([\sigma_2^2]\), the variance matrix of the model residual vector,

\[
[\sigma_2^2] = [U_2][\Sigma_2][U_2]^T \tag{14}
\]

in which, \([U_2]\) is the eigenvector matrix; \([\Sigma_2]\) is the eigenvalue matrix, a diagonal matrix, equation 13 means that the number of non zero eigenvalues of the matrix \([\sigma_2^2]\) is equal to the number of non zero eigenvalues of the matrix \([S_{ax}^{ARV}(\omega)]\). Therefore, the number of the uncorrelated phenomena in the time series \([x]\), as well as the number of the uncorrelated sources, can also be found by the eigenvalue decomposition of the matrix \([\sigma_2^2]\) in the time domain.

The number of non zero singular values of matrix \([\sigma_2^2]\) is the number of the uncorrelated phenomena in \([a_t]\), because \([\sigma_2^2]\) is the variance matrix of \([a_t]\). This means that the number of the uncorrelated phenomena in \([a_t]\) is equal to that in the time series \([x]\) and the number of the uncorrelated sources. Form the following matrix,

\[
[A(t)] = [(a_t)(a_{t+1}) \cdots (a_{t+m-1})]_{m \times m} \tag{15}
\]

in which \([a_t], (a_{t+1}), \ldots, (a_{t+m-1}), t = 1, \ldots, N - m,\) is a subset of \([a_t], t = 1, \ldots, N\). SVD of the matrix \([A(t)]\) will reveal the number of the uncorrelated phenomena in \([a_t]\), \([a_{t+1}], \ldots, (a_{t+m-1}), t = 1, \ldots, N - m\), and will also present some information about the number of uncorrelated phenomena in \([a_t]\). Because the number of uncorrelated phenomena in \([a_t]\) is certainly equal or larger than that in \([a_t], (a_{t+1}), \ldots, (a_{t+m-1}), t = 1, \ldots, N - m\).

When calculating the singular value decomposition of the matrix at \(t = 0, N - m\), and plotting the singular values as a function of time \(t\), the number of uncorrelated phenomena in \([a_t](a_{t+1}) \cdots (a_{t+m-1}), t = 1, \ldots, N - m\) can be determined by the number of significant singular values in the plot. The number of uncorrelated phenomena in \([a_t]\) is at least equal or larger that the number of significant singular values in the plot, as well as the number of uncorrelated phenomena in the time series and the number of uncorrelated sources. Then, the SVD of the matrix \([A(t)]\) can be served as a complimentary tool for the determination of the number of uncorrelated sources. It can be used to check the analysis results of the eigenvalue decomposition of \([S_{ax}^{ARV}(\omega)]\) and \([\sigma_2^2]\).

From above analysis, the whole time series analysis method for the determination of the number of uncorrelated sources can be summarized as following:

- acquisition of the reference signals;
- fit of the observations of the reference signals to obtain an ARV model;
- calculation of the cross spectral matrix \([S_{ax}^{ARV}(\omega)]\), the matrices \([A(t)]\) and \([\sigma_2^2]\) from the ARV model.
- eigenvalue decomposition of \([S_{ax}^{ARV}(\omega)]\) and \([\sigma_2^2]\) or singular value decomposition of \([A(t)]\).

In which, the most important is how to get an adequate model from the reference signals. The procedure to get an adequate model includes determination of the model order, model parameter estimation and the validity check of the model. Details about ARV model modelling can be found in reference [8].

5 Simulations

In this section, simulated data from an 8 DOFs linear vibration system was used to check the validity of the time series analysis method for the determination of the number of the uncorrelated sources. In the simulations, the inputs of the system represent the sources, and the outputs of the system represent the reference signals. In the analysis, only the outputs of the system are supposed to be available.

The system is a mechanical linear vibration system with viscous damping, whose main characteristic parameters are shown in Table 1. In the simulations, two excitations which are mutually uncorrelated were applied to the first and second DOF of the system. The all response data at 8 DOFs were calculated and used for the analysis. In order to check the validity of the time series analysis method clearly, all response data used in the analysis are free of noise. In the analysis, the response data are modelled by an ARV model with order 10 which has been checked to be adequate for all simulations presented in the paper. The analysis of the eigenvalue decomposition of \([S_{ax}^{ARV}(\omega)]\) and \([\sigma_2^2]\) or the singular value decomposition analysis of \([A(t)]\) are all presented. For comparison, the analysis results of the FFT-based PCA are also presented. In the analysis, only 1024 points response data from each DOF are used in the times series analysis method, but 16\times 1026 points of data are used in the FFT-based PCA.

In the first simulation, the input signals are all random signals. All analysis results are in Figure 2. The eigenvalues of the cross spectral matrix \([S_{ax}^{ARV}(\omega)]\) at each discrete frequency are plotted as a function of frequency, and are shown in 2.a.; the eigenvalues of the variance matrix of the model residual vector are shown in 2.d. Both indicate that two sources exist, and this conclusion is also confirmed by the singular values plot of the matrix \([A(t)]\) shown in 2.b. The analysis results of the FFT-based PCA shown in 2.c are not clear.

In the second example, one input is a random signal, and the second input is a swept sine signal(0-50Hz). The analysis results are shown in Figure 3. From the analysis results of the cross spectral matrix \([S_{ax}^{ARV}(\omega)]\) and the variance matrix of the model residual vector, it can be concluded that there are two uncorrelated sources. Again, this is confirmed by the analysis results from the matrix \([A(t)]\). The analysis results of the FFT-based PCA is not clear again.

In the third example shown in Figure 4, the first input re-
mains a random signal, but the second input is changed to a sine signal (157.5Hz). The analysis results from the cross spectral matrix \( S_{xy}(\omega) \) clearly shows that two uncorrelated sources exist, and the second uncorrelated source is a phenomenon at single frequency 157.5Hz. But the analysis results from the model residual variance matrix \( \sigma^2 \) and the matrix \( A(t) \) only reveal one source. It was found from a large amount of simulations that the analysis results from these two matrices are closely related to the energy distribution of the source over the frequency band. When the energy is distributed over a broad frequency band, the results from these two matrices will yield useful information, otherwise the results will fail to give correct information. Since the energy of second source in this example is concentrated around a narrow frequency band, the analysis results from the model residual variance matrix \( \sigma^2 \) and the matrix \( A(t) \) fail to detect this source.

6 Conclusions

From the analysis results in the above section, we can draw the following conclusions.

- Time series analysis modelling is an effective method for the estimation of the number of uncorrelated sources. The analysis results by the time series analysis method are less biased than that based on FFT, and free of the leakage errors. It is also suitable for the analysis of the phenomena with short duration.
- Eigenvalue decomposition of the cross spectral matrix obtained from the time series model is the most attractive method. It only needs one record of data, and its analysis results are the most easy to interpreted.
- The analysis results from the FFT-based PCA method are more difficult to interpret due to the leakage errors.
- Analysis results based on SVD of \( A(t) \) or on the eigenvalue decomposition of the model residual variance matrix \( \sigma^2 \), are closely related to the uncorrelated phenomena's energy distribution over the frequency band. For narrow energy distributions however (such as sine, swept sine with narrow frequency band), they fail to yield useful answers.

7 References


Table 1 Modal Parameters of 8 DOFs System

<table>
<thead>
<tr>
<th>mode</th>
<th>( f_i ) (Hz)</th>
<th>( \zeta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>0.01</td>
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<tr>
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<td>3</td>
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<td>5</td>
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<td>0.01</td>
</tr>
<tr>
<td>6</td>
<td>250</td>
<td>0.01</td>
</tr>
<tr>
<td>7</td>
<td>325</td>
<td>0.01</td>
</tr>
<tr>
<td>8</td>
<td>400</td>
<td>0.01</td>
</tr>
</tbody>
</table>

\( f_i \): \( i \)th natural frequency; 
\( \zeta_i \): \( i \)th damping ratio.
Figure 2: Analysis results with two random inputs
analysis results from (a) $S_{xy}(\omega)$; (b) $A(t)$; (c) PCA with FFT; (d) $[\alpha^2]$. 

(d) $S_{xy} = 7.4592689e+01 \quad 1.5396622e+01 \quad 6.5124954e-09 \quad 4.3140463e-09
2.4768383e-09 \quad 1.8559983e-09 \quad 1.1960959e-09 \quad 6.5009845e-10$

Figure 3: Analysis results with random input + swept sine
analysis results from (a) $S_{xy}^{ARV}(\omega)$; (b) $A(t)$; (c) PCA with FFT; (d) $[\alpha^2]$. 

(d) $S_{xy}^{ARV} = 1.1907087e+02 \quad 1.0092019e-01 \quad 6.5961995e-09 \quad 5.0954260e-09
3.8755051e-09 \quad 1.8559983e-09 \quad 1.1950959e-09 \quad 5.5009845e-10$
Figure 4: Analysis results with random input + sine
analysis results from (a) $S_{xx}^{ARV}(\omega)$; (b) $A(t)$; (c) PCA with FFT; (d) $\sigma_2^2$. 

(d) $S.V. = 2.6467152e+05$  $8.3892882e-03$  $5.7127679e-05$  $9.693738b-09$  
$3.5054493e-09$  $2.4146146e-09$  $1.2738662e-09$  $4.3806776e-10$