Identification of Modal Parameters for Defective Vibration Systems

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Abstract

A defective vibration system is a special linear vibration system which cannot be completely uncoupled by linear transformation. Previous studies have shown that the modal models of defective vibration systems are determined by modal parameters (modal frequencies, damping ratios and mode shape vectors), as well as the structural matrices of the vibration systems. So the whole identification of defective vibration systems includes two steps: (1) identification of the structure matrices; (2) estimation of the modal parameters. In this paper, the theory about the modal parameter identification of defective vibration systems by using the free responses of the systems is presented. The theory supposes the structure matrices of the systems are known, and uses the non-linear least square method for estimations of the modal parameters. Identification examples are also presented.

NOMENCLATURE

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1 Introduction

For a n DOF's linear vibration system with general viscous damping, its motion equation is

\[ \{ M \} \{ \ddot{z} \} + [D] \{ \dot{z} \} + [K] \{ z \} = \{ f(t) \} \]  

(1)

where \{ z \} is the displacement response vector of the system; \{ f(t) \} is the force vector; \{ M \}, \{ D \} and \{ K \} are the mass matrix, damping matrix and stiffness matrix of the system respectively. Let \{ y \} = \{ \{ z \} \, \{ \dot{z} \} \}. Then Eq. 1 can be written as

\[
\begin{bmatrix}
[D] & [M] \\
[M] & 0
\end{bmatrix}
\{ \dot{y} \} +
\begin{bmatrix}
[K] \\
0 & -[M]
\end{bmatrix}
\{ y \} =
\begin{bmatrix}
[I] \\
0
\end{bmatrix}
\{ f(t) \}
\]

(2)

if \{ M \} is nonsingular, by letting

\[ \{ A \} = -
\begin{bmatrix}
[D] & [M] \\
[M] & 0
\end{bmatrix}
\begin{bmatrix}
[K] \\
0 & -[M]
\end{bmatrix}
\]

\[ \{ B \} =
\begin{bmatrix}
[D] & [M] \\
[M] & 0
\end{bmatrix}
^{-1}
\begin{bmatrix}
[I] \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
[M]^{-1}
\end{bmatrix}
\]

\[ \{ C \} = \begin{bmatrix}
[I] \\
0
\end{bmatrix}
\]

Eq. 2 can be rewritten as

\[ \{ \dot{y} \} = \{ A \} \{ y \} + \{ B \} \{ f(t) \} \]  

(3)

\[ \{ z \} = \{ C \} \{ y \} \]  

(4)

Eqs. 3 and 4 are called the state equations of the vibration system, in which \{ A \} is the characteristic matrix, \{ B \} is the input matrix and \{ C \} is the output matrix. The eigenvalue decomposition of the matrix of \{ A \} gives

\[ \{ \Psi \}^{-1} \{ A \} \{ \Psi \} = \{ \Lambda \} \]  

(5)

where

\[ \{ \Lambda \} = \begin{bmatrix}
\lambda_1 & \cdots & \lambda_n
\end{bmatrix}
\]

\[ \{ \Psi \} = \begin{bmatrix}
\{ \psi \} & \{ \psi^* \}
\end{bmatrix}
\]

\[ \{ \psi \} = \{ \psi_1, \ldots, \psi_n \} \]

\[ \{ \psi^* \} = \{ \psi_n, \ldots, \psi_1 \} \]

in which \lambda_i, \lambda_i^*; i = 1, 2, \ldots, n are the eigenvalues of \{ A \}; \{ \Psi \} is the eigenvector matrix; \{ \psi \} is called the modal matrix of the system; * represents complex conjugate. Let \{ y \} = \{ \Psi \} \{ u \}. Eq. 3 can be rewritten as

\[ \{ \dot{u} \} = \{ \Lambda \} \{ u \} + \{ \Psi \}^{-1} \{ B \} \{ f(t) \} \]  

(6)

This is a matrix form of 2n one order differential equations. Therefore the response calculation of the system and the analysis of the system's dynamic properties become more easier than direct from Eq. 1. This is the Complex Modal Theory.

The complex modal theory is based on Eq. 5, which means that the characteristic matrix of the system, \{ A \} can be transformed to a diagonal matrix. The necessary and sufficient condition for a matrix which can be transformed to a diagonal matrix is that it must has a complete set of eigenvectors. In fact, not all matrices satisfy this condition. The appearance of multiple eigenvalues may cause a matrix has not a complete set of eigenvectors. The vibration system whose characteristic matrix has not a complete set of eigenvectors is called a defective vibration system, for which the Complex Modal Theory can not be applied. For this reason, a modal theory for defective systems has been developed[1], in which the concept of a structure matrix was introduced besides the modal parameters. Therefore the whole identification of defective vibration systems includes two steps: (1) identification of the structure matrix; (2) estimation of the modal parameters.

In this paper, the modal theory of defective vibration systems is shortly reviewed, and the theory about identification of the modal parameters of defective systems by using the free response is presented. The identification of the modal parameters for defective systems is totally different from that of non defective systems. It is assumed that the structure matrices of the systems must be known or have been identified. The procedure is more complex than that of non defective systems, in which the nonlinear least square method is used. Examples are also presented in the paper.

2 Modal theory of defective vibration systems[1]

If a vibration system has l pairs of distinct eigenvalues, its characteristic matrix \{ A \} can be transformed to a matrix below by similar transformation[2]

\[ \{ P \}^{-1} \{ A \} \{ P \} =
\]

\[ \begin{bmatrix}
[J_1(\lambda_1)] \\
& \ddots \\
& & [J_l(\lambda_l)]
\end{bmatrix}
\]

(7)

\[ [J_i(\lambda_i)] =
\]

\[ \begin{bmatrix}
[J_{i1}(\lambda_i)] \\
& \ddots \\
& & [J_{i\eta_i}(\lambda_i)]
\end{bmatrix}
\]

(8)

where \eta_i = n_{i1} \times n_{i1}.
and

\[ [J_{ij}(\lambda_i)] = \begin{bmatrix} \lambda_i & 1 & \cdots & 1 \\ \lambda_i & & & \cdots \\ & \ddots & & \vdots \\ & & \lambda_i & \cdots \\ & & & \lambda_i \end{bmatrix}_{n_i \times n_i} \]

(9)

\( l_i \) is the number of Jordan block matrices corresponding to the \( i \)th eigenvalue of the system; \( n_i \) is the multiplicity of \( i \)th eigenvalue; and \( n_{ij} \) is the dimension of \( j \)th Jordan block matrix corresponding to the \( i \)th eigenvalue; \( [P] \) is the eigenvector and the general eigenvector matrix. Eq.7 is the Jordan canonical form of matrix \([A]\). Besides the sequence of the Jordan block matrices in Eq. 8, this similar transformation is unique.

Supposing

\[ \lambda = \begin{bmatrix} \lambda_1[l]_{n_1 \times n_1} \\ \vdots \\ \lambda_l[l]_{n_l \times n_l} \end{bmatrix} \]

and introducing a matrix \([J]\) as below

\[ [J] = \begin{bmatrix} [J_1(\lambda_1)] - \lambda_1[I] \\ \vdots \\ [J_l(\lambda_l)] - \lambda_l[I] \end{bmatrix} \]

(10)

the free response function of the defective system can be deduced as

\[ \{x(t)\} = 2Re \left\{ (\phi)e^{\lambda t}\{u(0)\} \right\} + 2Re \left( \sum_{r=2}^{m} \frac{t^{r-1}}{(r-1)!}[\phi][J]^{r-2}e^{\lambda t}\{u(0)\} \right) \]

(11)

where \([\phi]\) and \(\{u(0)\}\) are defined in the following equation.

\[ [P] = \begin{bmatrix} [\phi] & [\phi^*] \\ [\phi^*] & [\phi] \end{bmatrix} \cdot \begin{bmatrix} \{u(0)\} \\ \{u^*(0)\} \end{bmatrix} = \begin{bmatrix} [\phi] \end{bmatrix}^{-1} \begin{bmatrix} \{x(0)\} \\ \{\dot{x}(0)\} \end{bmatrix} \]

\( m = \max(n_{ij}) \)

\([\phi]\) is called the modal matrix of the defective vibration system, which is determined by the eigenvectors and the general eigenvectors of the characteristic matrix \([A]\). \([\phi]\) can be written into block matrices corresponding to the distinct eigenvalues of the system.

\[ [\phi] = [[\phi_1], \cdots, [\phi_l]] \]

in which \([\phi^*]\) can be further divided into block matrices corresponding to the Jordan block matrices of each eigenvalue,

\[ [\phi_j] = [[\phi_{j1}], \cdots, [\phi_{ji}]] \]

where

\[ \{\phi_{j}^{(i)}\}_{i = 1, \cdots, l, j = 1, \cdots, l_i, k = 1, \cdots, n_{ij}} \text{ are called the modal shape vectors when they correspond to the eigenvectors of the matrix } [A], \text{ or the general modal shape vectors when they are related to the general eigenvectors.} \]

Let

\[ \begin{bmatrix} [L] \\ [L]^* \end{bmatrix} = [P]^{-1}[B] \]

The transfer function of the defective system can be acquired as

\[ [H(s)] = \sum_{r=1}^{m} \left( ([\phi][J]^{r-1}[s][I] - [\lambda])^{-1}[L] + [\phi^*][J]^{r-2}[s][I] - [\lambda^*])^{-1}[L]^* \right) \]

(12)

\([L]\) is called the modal participation matrix of the system.

For a linear vibration system

\[ \{X(s)\} = [H(s)][F(s)] \]

in which \(\{X(s)\}\) and \(\{F(s)\}\) are the Laplace transform of the displacement response and force vector of the system respectively. Substituting Eq. 12 into Eq. 13, and taking the inverse Laplace transform on the both side of Eq. 13, we can get the forced response of the defective vibration system.

\[ \{x(t)\} = 2Re \left( \sum_{r=2}^{m} \frac{t^{r-1}}{(r-1)!}[\phi][J]^{r-2}e^{\lambda t}\{u(0)\} \right) \]

(13)

By introducing the modal matrix, modal participation matrix and the structural matrix, the analytical formulae for the computation of the free response, forced response and transfer function of the defective vibration system is given out. This is the modal theory of defective vibration systems. For a non defective system, it satisfies

\[ [J] = 0 \]

(14)

(15)

In this situation, all formulae above change to the same forms as those in Complex Modal Theory. So Complex Modal Theory is a special case of the theory presented above.

3 Identification of the modal parameters

Above formulae have shown that the structure matrix plays an important role in the modal theory of defective vibration systems, as well as the modal parameters. Thus, the identification of a defective vibration system should include two steps:

1. identification of the structure matrix \([J]\),
2. identification of the modal parameters.
A method for the identification of the structure matrix has been developed in reference [3]. In this section, the theory of the identification of the modal parameters for defective systems by using the free response functions of the systems is presented. For the theory, it is assumed that the structure matrices of the systems are known or have been identified.

In Eq. 11

\[
e^{\lambda_1 t} [u(0)] \\
e^{\lambda_2 t}[u_{n-1}(0)] \\
\vdots \\
e^{\lambda_n t}[u_n(0)]
\]

\[
\begin{bmatrix}
  u_1(0) \\
  u_2(0) \\
  \vdots \\
  u_n(0)
\end{bmatrix}
\]

(16)

Then

\[
\lambda_i = \alpha_i + j \beta_i
\]

(19)

Then

\[
(e^{\lambda_i t})_{n \times n_1} = e^{\alpha_i t} \cos \beta_i t + j e^{\alpha_i t} \sin \beta_i t
\]

(20)

Substituting Eqs. 17, 18 and 20 into Eq. 11 gives

\[
\{e^{\lambda_i t}\}_{n \times n_1} = \{e^{\alpha_i t} \cos \beta_i t\}_{n_1 \times n_1} + j \{e^{\alpha_i t} \sin \beta_i t\}_{n_1 \times n_1}
\]

(21)

in which

\[
\{e^{\alpha_i t} \cos \beta_i t\}_{n_1 \times n_1} = \begin{bmatrix}
  e^{\alpha_i t} \cos \beta_i t_{1,1} & \cdots & e^{\alpha_i t} \cos \beta_i t_{n_1,1} \\
  \vdots & \ddots & \vdots \\
  e^{\alpha_i t} \cos \beta_i t_{1,n_1} & \cdots & e^{\alpha_i t} \cos \beta_i t_{n_1,n_1}
\end{bmatrix}
\]

\[
\{e^{\alpha_i t} \sin \beta_i t\}_{n_1 \times n_1} = \begin{bmatrix}
  e^{\alpha_i t} \sin \beta_i t_{1,1} & \cdots & e^{\alpha_i t} \sin \beta_i t_{n_1,1} \\
  \vdots & \ddots & \vdots \\
  e^{\alpha_i t} \sin \beta_i t_{1,n_1} & \cdots & e^{\alpha_i t} \sin \beta_i t_{n_1,n_1}
\end{bmatrix}
\]

Rewrite the matrix \([\varphi']\) into \(l\) block matrices corresponding to the distinct eigenvalues

\[
[\varphi'] = \begin{bmatrix}
  [\varphi'_{\lambda_1}] & [\varphi'_{\lambda_2}] & \cdots & [\varphi'_{\lambda_l}]
\end{bmatrix}
\]

(22)

and let

\[
\{\alpha^{i,j}\} = \sum_{k=1}^{n_i} 2Re\{\varphi^{j,k}_{\lambda_i}\}
\]

(23)

in which \(\{\varphi^{j,k}_{\lambda_i}\}\) is the \(k\)th column vector of matrix \(\{\varphi'_{\lambda_i}\}\). Then Eq. 21 can be written as

\[
\{z(t)\} = \sum_{i=1}^{m_1} \sum_{j=1}^{n_i} \left( \{\alpha^{i,j}\} t^i e^{-\alpha_i t} \cos \beta_i t - \{\beta^{i,j}\} t^i e^{-\alpha_i t} \sin \beta_i t \right)
\]

(25)

in which

\[
m_i = \max \{n_1, \ldots, n_l\}
\]

(26)

Eq. 25 is the model for the modal parameter identification of the defective system. Let \(\{\theta\}\) represents all parameters which are needed to be identified.

\[
\{\theta\} = \{\{\alpha^{i,j}\}, \{\beta\}, a_i, b_i : j = 1, \ldots, m_i, i = 1, \ldots, l\}
\]

(27)

The model can be written as the following short form

\[
\{z(t)\} = f(\{\theta\}, t)
\]

(28)

As the function \(f(\{\theta\}, t)\) being a nonlinear function of the parameters \(\{\theta\}\), nonlinear square method must be used. In this paper, we used the Levenberg-Marquardt nonlinear least square method.[4].

Suppose \(\{z'(t)\}\) is the measurement of the free response of the system, and let

\[
\{g(\{\theta\})\} = \left\{ \begin{array}{c}
  \{z'(t_1)\} - f(\{\theta\}, t_1) \\
  \vdots \\
  \{z'(t_N)\} - f(\{\theta\}, t_N)
\end{array} \right\}
\]

(29)

where \(N\) is the number of observations of the free response. Then the recursive estimation formula of \(\{\theta\}\) is

\[
\{\hat{\theta}^{i+1}\} = \{\hat{\theta}^i\} - \{v_i[I + [R_i]^T[R_i]]^{-1}[R_i]^T[g(\{\hat{\theta}^i\})]\}
\]

(30)

in which \(\{\hat{\theta}^i\}\) is the estimation of the parameters at the \(i\)th recursive, \(v_i\) is the damping factor, and \([R_i]\) is the Jacobian matrix.

\[
[R_i] = \begin{bmatrix}
  \frac{\partial f(\{\theta\}, t_1)}{\partial \theta_1} & \cdots & \frac{\partial f(\{\theta\}, t_1)}{\partial \theta_{N,OP}} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial f(\{\theta\}, t_N)}{\partial \theta_1} & \cdots & \frac{\partial f(\{\theta\}, t_N)}{\partial \theta_{N,OP}}
\end{bmatrix}
\]

(31)

where \(N,\text{OP}\) is the number of the parameters in \(\{\theta\}\), and \(\theta_i\) is the \(i\)th element in \(\{\theta\}\).

For the nonlinear least square method, the most important thing is the choice of initial values of the parameters. Improper choice of initial values may cause the local minimum problem, and yield incorrect estimation. In our estimation of the modal parameters, it is assumed that the structure matrix has been identified. In the identification of the structure matrix, the modal frequencies and damping ratios of the system have been identified as well. The initial value
of \(a_i, b_i\) can be obtained from the estimation of the modal frequencies and the damping ratio.

\[
\begin{align*}
\frac{\omega_i}{\omega_i} &= -\frac{\omega_i}{\omega_i} \\
\frac{\zeta_i}{\zeta_i} &= \sqrt{1 - \frac{\omega_i}{\omega_i}}
\end{align*}
\]

(32) (33)

Substituting the initial value of \(a_i, b_i\) into Eq. 25 and solving the linear least square problem, we can get the initial values of \(\{\alpha^{(j)}\}\) and \(\{\beta^{(j)}\}\). When the parameters in Eq. 25 have been estimated by nonlinear least square method, the modal parameters of the system can then be identified from the parameters \(\{\alpha^{(j)}\}, \{\beta^{(j)}\}, a_i\) and \(b_i\). The modal frequency and damping ratio can be identified by following equations.

\[
\begin{align*}
\omega_i &= \sqrt{\alpha_i^2 + \beta_i^2} \\
\zeta_i &= \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}}
\end{align*}
\]

(34) (35)

The modal vector and the general modal vector can be identified from the parameters \(\{\alpha^{(j)}\}, \{\beta^{(j)}\}\), but the procedure is very complex, and it will be discussed in detail in the following part.

First we give out two theorems which are useful in the modal shape vectors and general modal shape vectors identification[5].

**Theorem 1** If \(\{p_1\}, \{p_2\}, \ldots, \{p_n\}\) are the eigenvector and the general eigenvectors of the matrix \([A]\) corresponding to the eigenvalue \(\lambda_1\), they satisfy the following equations.

\[
\begin{align*}
[\lambda_i[I] - [A]] [p_1] &= 0 \\
[\lambda_i[I] - [A]] [p_j] &= -[p_{j-1}] \\
&j = 2, 3, \ldots, n
\end{align*}
\]

(36) (37)

Let

\[
\begin{align*}
\{p_1\} &= \gamma_1 \{p_1\} \\
\{p_2\} &= \gamma_2 \{p_1\} + \gamma_1 \{p_2\} \\
&\ldots \ldots \\
\{p_n\} &= \gamma_n \{p_1\} + \ldots + \gamma_1 \{p_n\}
\end{align*}
\]

Then \(\{p_1\}, \ldots, \{p_n\}\) are also the eigenvector and the general eigenvectors corresponding to the eigenvalue \(\lambda_1\).

**Theorem 2** If \(\{p_1\}, \{p_2\}, \ldots, \{p_n\}\) are the eigenvector and the general eigenvectors of the matrix \([A]\) corresponding to the first Jordan block matrix \(\lambda_i\), \(\{\phi_1^{(j)}\}, \{\phi_2^{(j)}\}, \ldots, \{\phi_n^{(j)}\}\) are the eigenvector and the general eigenvectors corresponding to the second Jordan block matrix, and \(n_1 \geq n_2\), they satisfy the following equations.

\[
\begin{align*}
[\lambda_i[I] - [A]] [p_1] &= 0 \\
[\lambda_i[I] - [A]] [p_j] &= -[p_{j-1}] \\
&j = 2, 3, \ldots, n_1 \\
[\lambda_i[I] - [A]] [p_j] &= -[p_{j-1}] \\
&j = 2, 3, \ldots, n_2
\end{align*}
\]

(38) (39)

Let

\[
\begin{align*}
\{p_1\} &= \{p_1\} \\
\{p_2\} &= \{p_1\} + \epsilon \{p_2\} \\
&\ldots \ldots \\
\{p_n\} &= \{p_1\} + \epsilon \{p_n\}
\end{align*}
\]

in which \(\epsilon\) is an arbitrary number. Then \(\{p_1\}, \ldots, \{p_n\}\) are also the eigenvector and the general eigenvectors corresponding to the first Jordan block matrix. And let

\[
\begin{align*}
\{p_1\} &= \{p_1\} + \epsilon \{p_1\} \\
\{p_2\} &= \{p_1\} + \epsilon \{p_2\} \\
&\ldots \ldots \\
\{p_n\} &= \{p_1\} + \epsilon \{p_n\}
\end{align*}
\]

Then \(\{p_1\}, \ldots, \{p_n\}\) are also the eigenvector and the general eigenvectors corresponding to the second Jordan block matrix.

The proof of these two theorems is very simple, just by substituting the eigenvectors and the general eigenvectors into Eqs. 35 and 36, and finding out that the eigenvectors and the general eigenvectors all satisfy the equations. For the modal shape vectors and the general modal shape vectors which are the sunsets of the eigenvectors and the general eigenvectors, they also satisfy above two theorems.

Now let us discuss some cases respectively.

(1) Suppose the defective system has only one distinct eigenvalue \(\lambda_1 = a_1 + j\beta_1\) and only one first Jordan block matrix corresponding to the eigenvalue. Then Eq. 25 can be rewritten as

\[
x(t) = \sum_{j=1}^{n_1} \left( (\alpha^{(j)}_1) t e^{\alpha^{(j)}_1 t} \cos b_1 t - (\beta^{(j)}_1) t e^{\alpha^{(j)}_1 t} \sin b_1 t \right)
\]

(38)

Let

\[
\begin{align*}
\{\phi_1^{(1)}\} &= \{\alpha^{(1)}_1\} + j\{\beta^{(1)}_1\} \\
\{\phi_2^{(1)}\} &= \{\alpha^{(1)}_1\} + j\{\beta^{(1)}_1\} \\
&\ldots \ldots \\
\{\phi_n^{(1)}\} &= \{\alpha^{(1)}_1\} + j\{\beta^{(1)}_1\}
\end{align*}
\]

According to Theorem 1, \(\{\phi_1^{(1)}\}, \ldots, \{\phi_n^{(1)}\}\) are the estimation of the modal shape vector and the general modal shape vectors.
(2) Suppose the defective system has only one distinct eigenvalue \( \lambda_1 = a_1 + jb_1 \) but two Jordan block matrices, \( n_{12} \times n_{12} \) and \( n_{11} \times n_{11} \) corresponding to the eigenvalue, where \( n_{11} \geq n_{12} \). For this case, the model for the parameter identification is the same as Eq.37. In order to identify all the modal shape vectors and the general modal shape vectors of the system, the free responses under two uncorrelated initial states of the system are needed. By using the free response of the system under the first initial condition, we can estimate another set of the parameters \( a_1, b_1, \{a_1^{(j)}\}, \{b_1^{(j)}\} \) from which we can get

\[
\{\phi_1^{1,i}\} = \left\{ (\alpha_1^{1,n_{11}-i+1}) + j(\beta_1^{1,n_{11}-i+1}) \right\} \times (n_{11} - i)
\]

\[
\{\phi_1^{1,n_{11}}\} = \{a_1^{1,1}\} + j\{b_1^{1,1}\}
\]

Using the free response of the system under the second initial condition, we can also estimate the parameters \( a_1, b_1, \{a_1^{(j)}\}, \{b_1^{(j)}\} \) from which we can get

\[
\{\phi_2^{1,i}\} = \left\{ (\alpha_1^{1,n_{12}-i+1}) + j(\beta_1^{1,n_{12}-i+1}) \right\} \times (n_{12} - i)
\]

\[
\{\phi_2^{1,n_{12}}\} = \{a_1^{1,1}\} + j\{b_1^{1,1}\}
\]

According to Theorem 2, \( \{\phi_1^{1,i}\}, i = 1, \ldots, n_{11}, \{\phi_2^{1,i}\}, i = 1, \ldots, n_{12} \) are the estimations of all the eigenvectors and the general eigenvectors of the defective system. When the defective system has only one distinct eigenvalue but \( n_1 \) Jordan block matrices corresponding to the eigenvalue, in order to identify all the modal shape vectors and the general modal shape vectors of the system, the free responses under \( n_1 \) uncorrelated initial states of the system are needed. From the free response under one uncorrelated initial condition, a set of the modal shape vector and the general modal shape vectors corresponding to one Jordan block matrix can be identified. However, please make sure that the \( n_1 \) initial conditions must be uncorrelated. Otherwise, the same identified results of the modal shape vectors and the general modal shape vectors are correlated, and thus the modal shape vectors and the general modal shape vectors are not fully identified.

(3) For the system with many distinct eigenvalues, the procedure in (1) or (2) can be applied to each distinct eigenvalue, and all the modal shape vectors and the general modal shape vectors of the defective system can then be identified.

4 Example

In this part, simulated free responses of two defective vibration systems are used to check the correctness of the theory presented in the paper. The first one is a 2 DOFs linear system, and the second one is a four DOFs linear system. Both systems are with viscous damping.

In the first system, the \([M], [D]\) and \([K]\) matrices are

\[
[M] = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, [D] = \begin{bmatrix} 8.0 & -4.0 \\ -3.0 & 8.0 \end{bmatrix}, [K] = \begin{bmatrix} 75.0 & 0.0 \\ 0.0 & 100.0 \end{bmatrix}
\]

The free response of the system is simulated on the computer under the initial condition \( z_1(0) = 1.0, z_2(0) = 0.0, x_1(0) = 0.0 \) and \( x_2(0) = 0.0, 5\% \) random noise is added to the response. In the analysis, the free response is first analyzed by Eigensystem Realization Algorithm (ERA). Result shows that there are two repeated modes in the free response system. Then the data is analyzed by the nonlinear least square method (NLS). Final analysis results are shown in Tab.1. Results show that both ERA and NLS yield correct results, however only NLS can give the estimation of the general modal shape vector.

In the second system, the \([M], [D]\) and \([K]\) matrices are

\[
[M] = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}
\]

\[
[D] = \begin{bmatrix} 9.4393 & -2.1213 & -6.5607 & -8.0000 \\ -0.6213 & 8.0000 & 3.6213 & 8.0000 \\ -4.4393 & 2.1213 & 11.5607 & -8.0000 \\ 1.5600 & 0.0000 & 15.0000 & 19.0000 \end{bmatrix}
\]

\[
[K] = \begin{bmatrix} 62.5000 & 0.0000 & -37.5000 & -75.0000 \\ 0.0000 & 25.0000 & 0.0000 & 75.0000 \\ -37.5000 & 0.0000 & 62.5000 & -75.0000 \\ 0.0000 & 0.0000 & 0.0000 & 100.0000 \end{bmatrix}
\]

The free responses of the system under two initial conditions \( x_1(0) = 1.0, x_2(0) = 0.0, x_3(0) = 0.0, x_4(0) = 0.0, z_1(0) = 0.0, z_2(0) = 0.0, z_3(0) = 0.0, z_4(0) = 0.0, x_1(0) = 1.0, x_2(0) = -2.0, x_3(0) = 0.0, x_4(0) = 0.0, z_1(0) = 0.0, z_2(0) = 0.0, z_3(0) = 0.0, z_4(0) = 0.0 \) are the same as the theoretical values (modal frequency: 1.1254Hz, damping ratio: 0.8485). The identified
modal shape vectors and the general modal shape vectors are shown in Tab.2. To check the identified modal shape vectors and the general modal shape vectors, the characteristic matrix of the system is formed from \([M],[D]\) and \([K]\) matrices, and the eigenvector matrix \([P]\) is calculated from the identified modal vectors and the general modal vectors, and then \([P^{-1}[A][P]\) is calculated. The result is show in Tab.3, which is a Jordan canonical matrix. That means the identified modal vectors and the general modal vectors are correct.

5 Conclusion

The identification procedure for a defective vibration system includes two aspects; (1) identification of the system matrix; (2) identification of the modal parameter, because the speciality of the defective vibration system. In this paper, theory about the modal parameter identification for defective vibration systems is presented, which uses the free response of the system and is based on the known structure matrix of the system. Simulated examples show that the theory can correctly identify the all modal parameters of the defective vibration system, though it is much more complex when compared with the non defective system.

6 References


Tab. 1 Identified results for the Two DOFs System

<table>
<thead>
<tr>
<th></th>
<th>theoretical</th>
<th>ERA</th>
<th>NLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_c$</td>
<td>1.1254</td>
<td>1.1338</td>
<td>1.1043</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.8485</td>
<td>0.8541</td>
<td>0.8553</td>
</tr>
<tr>
<td>modal</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>vector</td>
<td>-0.3333 + j0.6236</td>
<td>-0.3774 + j0.6113</td>
<td>-0.3546 + j0.6761</td>
</tr>
<tr>
<td>general</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>mod. vec.</td>
<td>-0.0734 + j0.4740</td>
<td>-0.0914 + j0.4849</td>
<td></td>
</tr>
</tbody>
</table>

Tab. 2 Identified results for the Four DOFs system
(the modal shape vectors and the general modal shape vectors)

<table>
<thead>
<tr>
<th>$\phi_1^{-1}$</th>
<th>$\phi_2^{-1}$</th>
<th>$\phi_3^{-1}$</th>
<th>$\phi_4^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000 + j0.0000</td>
<td>0.1699 + j0.1429</td>
<td>-0.7133 + j0.5320</td>
<td>-0.1108 + j0.0304</td>
</tr>
<tr>
<td>-1.0000 + j0.0000</td>
<td>-0.1698 - j0.1429</td>
<td>1.1385 - j0.3454</td>
<td>0.2622 + j0.1800</td>
</tr>
<tr>
<td>1.0000 + j0.0000</td>
<td>0.1699 + j0.1429</td>
<td>-0.5175 + j0.436</td>
<td>0.0197 + j0.0827</td>
</tr>
<tr>
<td>-0.2000 - j0.7484</td>
<td>0.2314 + j0.1676</td>
<td>0.3384 + j0.4029</td>
<td>-0.1390 - j0.1306</td>
</tr>
</tbody>
</table>

Tab. 3 Check of the Identified results for the Four DOFs System

$[P]^{-1}[A][P] =$

\[
\begin{bmatrix}
-6.0002 + j3.7415 & 1.0005 - j0.0004 & -0.0003 - j0.0003 & 0.0006 - j0.0002 \\
0.0000 + j0.0000 & -5.9998 + j3.7418 & 0.0000 + j0.0000 & 0.0003 + j0.0003 \\
0.0000 + j0.0000 & 0.0006 - j0.0003 & -5.9997 + j3.7412 & 1.0006 + j0.0005 \\
0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & -6.0003 + j3.7421 \\
0.0001 + j0.0003 & 0.0009 - j0.0003 & 0.0009 - j0.0004 & 0.0001 + j0.0011 \\
0.0011 + j0.0000 & -0.0017 - j0.0010 & -0.0004 - j0.0012 & 0.0013 - j0.0000 \\
0.0000 + j0.0000 & 0.0009 - j0.0005 & 0.0004 - j0.0007 & 0.0007 + j0.0008 \\
0.0000 + j0.0000 & -0.0025 - j0.0010 & -0.0022 + j0.0004 & 0.0006 - j0.0025 \\
0.0001 - j0.0003 & 0.0009 + j0.0003 & 0.0009 + j0.0004 & 0.0001 - j0.0011 \\
0.0011 + j0.0000 & -0.0017 + j0.0010 & -0.0004 + j0.0012 & 0.0012 + j0.0004 \\
0.0000 + j0.0000 & 0.0009 + j0.0005 & 0.0004 + j0.0007 & 0.0007 - j0.0008 \\
0.0000 + j0.0000 & -0.0024 + j0.0010 & -0.0022 - j0.0004 & 0.0006 + j0.0025 \\
-6.0002 - j3.7415 & 1.0005 + j0.0004 & -0.0003 + j0.0003 & 0.0006 - j3.7421 \\
0.0000 + j0.0000 & -5.9998 - j3.7418 & 0.0000 + j0.0000 & 0.0003 - j0.0003 \\
0.0000 + j0.0000 & 0.0006 + j0.0003 & -5.9997 - j3.7412 & 1.0006 - j0.0005 \\
0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & -6.0003 - j3.7421
\end{bmatrix}
\]

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