Modal Parameter Estimation:
A Unified Matrix Polynomial Approach

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ABSTRACT

As many different modal parameter estimation algorithms have evolved over the past twenty years, the ability to view all algorithms from a unified perspective becomes important in order to compare and contrast the numerical characteristics of each algorithm. An approach that recognizes the underlying matrix polynomial structure of the numerical algorithms is one approach that achieves this goal. The following development serves as a framework to most modal parameter estimation algorithms that have been documented in the literature. The overall approach is given as well as the fundamental or basic equation formulation for each method. While there are many implementations of these algorithms in use, that may or may not utilize the theoretical approach given here, the ability to view each method in a common framework allows the numerical characteristics of each algorithm to be identified as advantages and disadvantages of the various methods. Note that the practical or commercial implementation of these algorithms may be presented differently elsewhere in order to reflect the specific numerical solution utilized.

1. Introduction

Modal parameter estimation is a special case of system identification where the a priori model of the system is known to be in the form of modal parameters. Over the past twenty years, a number of algorithms have been developed to estimate modal parameters from measured frequency or impulse response function data. Algorithms are reformulated using a common mathematical structure. This reformulation attempts to characterize different classes of modal parameter estimation techniques in terms of the structure of the underlying matrix polynomials rather than the physically based models used historically. Since the modal parameter estimation process involves a greatly over-determined problem (more data than independent equations), this reformulation is helpful in understanding the different numerical characteristics of each algorithm and, therefore, the slightly different estimates of modal parameters that each algorithm yields. As a part of this reformulation of the algorithms, this work has focussed on the development of a conceptual understanding of modal parameter estimation technology. This understanding involves the ability to conceptualize the measured data in terms of the concept of Characteristic Space, the data domain (time, frequency, spatial), the evaluation of the order of the problem, the condensation of the data, and a common parameter estimation theory that can serve as the basis for developing any of the algorithms in use today. The following sections will review these concepts as applied to the current modal parameter estimation methodology.

2. Definition: Modal Parameters

Modal identification involves estimating the modal parameters of a structural system from measured input-output data. Most current modal parameter estimation is based upon the measured data being the frequency response function or the equivalent impulse response function, typically found by inverse Fourier transforming the frequency response function. Modal parameters include the complex-

| Table 1: Acronyms - Modal Parameter Estimation Algorithms |
|-------------|------------------|
| CEA         | Complex Exponential Algorithm [1,2] |
| 1.SCE       | Least Squares Complex Exponential [1,2] |
| PTD         | Polynreference Time Domain [3,4,5] |
| ITD         | Ibrahim Time Domain [6] |
| MRITD       | Multiple Reference Ibrahim Time Domain [9] |
| ERA         | Eigensystem Realization Algorithm [13,14] |
| PFD         | Polyreference Frequency Domain [9,19,21,24,33] |
| SFD         | Simultaneous Frequency Domain [3] |
| MRFD        | Multi-Reference Frequency Domain [4] |
| RFP         | Rational Fraction Polynomial [56] |
| OP          | Orthogonal Polynomial [25,26,29] |
| CMIF        | Complex Mode Indication Function [124] |

501
Modal participation vectors are a result of multiple reference modal parameter estimation algorithms and define how each modal vector is excited from each of the reference locations included in the measured data. The combination of the modal participation vector \( (L_r) \) and the modal vector \( (\psi_r) \) for a given mode give the residue matrix \( (A_{yp} = L_{yp}\psi_{yp}) \) for that mode. In general, these two vectors represent portions of the right and left eigenvectors associated with the structural system for that specific mode of vibration. Normally, the system can be assumed to be reciprocal and the right and left eigenvectors, and therefore the modal participation vector and the modal vector, will be proportional to one another. Under this assumption, the modal participation vector can be used in a weighted least squares error solution procedure to estimate the modal vectors in the presence of multiple references. Theoretically, for reciprocal systems, these modal participation factors should be in proportion to the modal coefficients of the reference degrees of freedom for each modal vector.

In general, modal parameters are considered global properties of the system. The concept of global modal parameters simply means that there is only one answer for each modal parameter and that the modal parameter estimation solution procedure enforces this constraint. Every frequency response or impulse response function measurement theoretically contains the information that is represented by the characteristic equation, the modal frequencies and damping. If individual measurements are treated in the solution procedure independent of one another, the solution procedure does not guarantee that a single set of modal frequencies and damping will be generated. In likewise manner, if more than one reference is measured in the data set, redundant estimates of the modal vectors can be estimated unless the solution procedure utilizes all references in the estimation process simultaneously. Most of the current modal parameter estimation algorithms estimate the modal frequencies and damping in a global sense but very few estimate the modal vectors in a global sense by enforcing the reciprocity constraint.

3. Modal Identification Concepts

The current approach in modal identification involves using numerical techniques to separate the contributions of individual modes of vibration in measurements such as frequency response functions. The concept involves estimating the individual single degree of freedom (SDOF) contributions to the multiple degree of freedom (MDOF) measurement.

\[
[H(\omega)]_{k_r \times k_e} = \sum_{j=1}^{N} \frac{A_r \psi_{rj} \psi_{je}}{j\omega - \lambda_r} + \frac{A_r^* \psi_{rj} \psi_{je}}{j\omega - \lambda_r^*}
\]

This concept is mathematically represented in Equation (1) and graphically represented in Figures 2 and 3.
3.1 Concept: Model Order Relationships

The estimation of an appropriate model order is the most important problem encountered in modal parameter estimation. This problem is complicated due to the formulation of the parameter estimation model in the time or frequency domain, due to a single or multiple reference formulation of the modal parameter estimation model, and the effects of random and bias errors on the modal parameter estimation model. The basis of the formulation of the correct model order can be seen by expanding the second order matrix equation of motion to a higher order model (2N).

This is a necessary process required to handle the case where the spatial information (measured degrees of freedom) is truncated to a size smaller than the number of modal frequencies in the measured data. There are several ways that this concept can be developed. One method is to start with the steady-state matrix equation of motion and, for example, Laplace transform this equation to the s domain.

\[
\begin{bmatrix}
| M | s^2 + | C | s + | K |
\end{bmatrix}
\begin{bmatrix}
X(s)
\end{bmatrix}
= \{ F(s) \} \tag{2}
\]

The above matrix equation yields a characteristic matrix polynomial equation in the following form:

\[
| M | s^2 + | C | s + | K |
= 0 \tag{3}
\]

This characteristic equation is a matrix polynomial of second order which can be further partitioned as follows:

\[
\begin{bmatrix}
| M_{11} | & \cdots & | M_{1n} | \\
| M_{21} | & \cdots & | M_{2n} | \\
\vdots & \ddots & \vdots \\
| M_{n1} | & \cdots & | M_{nn} |
\end{bmatrix}
\begin{bmatrix}
| s^2 | & \cdots & | s | & 1 \\
\vdots & \ddots & \vdots & \vdots \\
| s | & \cdots & | 1 |
\end{bmatrix}
= 0
\]

This partitioned equation can be expanded to a higher order (2N) matrix polynomial and put in a generic form as follows:

\[
| \Gamma_2N | s^{2N} + | \Gamma_{2n-1} | s^{2n-1} + | \Gamma_{2n-2} | s^{2n-2} + \cdots + | \Gamma_0 | = 0 \tag{4}
\]

Note that the size of the matrices [\Gamma] is the same as the size of the partitioned submatrices in the previous equation. Also note that each [\Gamma] matrix involves a matrix product and summation of several [M], [C], and [K] submatrices. The roots of this characteristic matrix polynomial equation are the same as the original second order matrix polynomial equation.

The limit of this process would be to reduce the size of the matrices to a scalar equation with high model order (2N).

\[
| a_{2N} s^{2N} + a_{2N-1} s^{2N-1} + a_{2N-2} s^{2N-2} + \cdots + a_0 |
= 0 \tag{5}
\]

Likewise, the roots of this scalar characteristic equation are the same as the original second order matrix polynomial equation.

Therefore, the number of characteristic values (number of modal frequencies, number of roots, number of poles, etc.) that can be determined depends upon the size of the matrix coefficients involved in the model and the order of the polynomial terms in the model. There are a significant number of procedures that have been formulated particularly for aiding in these decisions and selecting the appropriate estimation model. Much research effort has been expended over the last five years on this effort. Procedures for estimating the appropriate matrix size and model order are another of the differences between various estimation procedures.

3.2 Concept: Fundamental Measurement Models

Most current modal parameter estimation algorithms utilize frequency or impulse response functions as the data, or known information, to solve for modal parameters. The general equation that
can be used to represent the relationship between the measured frequency response function matrix and the modal parameters is shown in Equations (7) and (8).

\[ H(\omega) = \begin{bmatrix} \psi \end{bmatrix}_{N \times 2N} \begin{bmatrix} \frac{1}{j\omega - \lambda_i} \end{bmatrix}_{2N \times 2N} L \begin{bmatrix} \psi \end{bmatrix}^T_{2N \times N_i} \]  

(7)

\[ H(\omega)^T = \begin{bmatrix} L \end{bmatrix}_{N \times 2N} \begin{bmatrix} \psi \end{bmatrix}_{2N \times N_i}^T \]  

(8)

Impulse response functions are rarely directly measured but are calculated from associated frequency response functions via the inverse FFT algorithm. The general equation that can be used to represent the relationship between the impulse response function matrix and the modal parameters is shown in Equations (9) and (10).

\[ h(t) = \begin{bmatrix} \psi \end{bmatrix}_{N \times 2N} \begin{bmatrix} e^{j\lambda t} \end{bmatrix}_{2N \times 2N} L \begin{bmatrix} \psi \end{bmatrix}^T_{2N \times N_i} \]  

(9)

\[ h(t)^T = \begin{bmatrix} L \end{bmatrix}_{N \times 2N} \begin{bmatrix} \psi \end{bmatrix}_{2N \times N_i}^T \]  

(10)

Many modal parameter estimation algorithms have been originally formulated from Equations (7) through (10). However, a more general development for all algorithms is based upon relating the above equations to a general matrix polynomial approach.

### 3.3 Concept: Characteristic Space

From a conceptual viewpoint, the measurement space of a modal identification problem can be visualized as occupying a volume with the coordinate axes defined in terms of the three sets of characteristics. Two axes of the conceptual volume correspond to spatial information and the third axis to temporal information. The spatial coordinates are in terms of the input and output degrees of freedom (DOFs) of the system. The temporal axis is either time or frequency depending upon the domain of the measurements. These three axes define a 3-D volume which is referred to as the Characteristic Space as noted previously in Figure 1.

This space or volume represents all possible measurement data as expressed by Equations (7) through (10). This conceptual representation is very useful in understanding what data subspace has been measured. Also, this conceptual representation is very useful in recognizing how the data is organized and utilized with respect to different modal parameter estimation algorithms. Information parallel to one of the axis consists of a solution composed of the superposition of the characteristics defined by that axis. The other two characteristics determine the scaling of each term in the superposition.

Any structural testing procedure measures a subspace of the total possible data available. Modal parameter estimation algorithms may then use all of this subspace or may choose to further limit the data to a more restrictive subspace. This is not a problem since it is theoretically possible to estimate the characteristics of the total space by measuring a subspace which samples all three characteristics. Experimentally, any point in this space can be measured. The particular subspace which is measured and the weighting of the data within the subspace in an algorithm are the main differences between the various modal identification procedures which have been developed historically.

In general, the amount of information in a measured subspace greatly exceeds that which is necessary to solve for the unknown parameters. Another major difference between the various modal parameter estimation procedures is the type of condensation algorithms that are used to reduce to the data to match the number of unknowns (for example, least squares (LS), singular value decomposition (SVD), etc.). As is the case with any over-specified solution procedure, there is no unique answer. The answer that is obtained depends upon the data that is selected, the weighting of the data and the unique algorithm used in the solution process. Historically, this point has created some confusion since many users expect different methods to give exactly the same answer.

An important realization is that various modal identification algorithms use a different data subspace. As might be expected, the selection of the subspace has a significant influence on the results. In order to estimate all of the modal parameters, the subspace must encompass a region which includes contributions from all three characteristics. A classic example is the necessity to use multiple reference (inputs and outputs) data in order to estimate repeated roots.

In the past, many modal techniques used information (data subspace) where only one or two characteristics were varied. For example, in the early 1970's, the modal parameter estimation methods of that era only fit one unit impulse function or one frequency response at a time. In this case, only the temporal characteristic is used and as might be expected only temporal characteristics (modal frequencies) can be estimated from the single measurement. In practice, multiple measurements were taken and the spatial information was extracted from the multiple measurements by successive estimation of the residues for each mode from each measurement.

As the sophistication of data acquisition equipment and modal parameter estimation algorithms evolved, later techniques fit the data in a plane of the characteristic space. For example, this corresponds to the data taken at a number of response points but from a single excitation point or reference. This representation of a column of measurements is shown as one of the planes of information in Figure 4. For this situation, which represents a single input reference, it is not possible to compute repeated roots and it is difficult to separate closely coupled modes due to the lack of spatial data. Many modal identification algorithms utilize data taken at a large number of output DOFs due to excitation at a small number of input DOFs. Data taken in this manner is consistent with a multi-excitation type of test. The conceptual representation of several columns of this potential measurement matrix is shown in Figure 4.
Some modal identification algorithms utilize data taken at a large number of input DOFs and a small number of output DOFs. Data taken in this manner is consistent with a roving hammer type of excitation with several fixed output sensors. This data can also be generated by transposing the data matrix acquired using a multiexcitor test. The conceptual representation of several rows of this potential measurement matrix is shown in Figure 5.

A small number of modal identification algorithms utilize data taken at a large number of output DOFs and input DOFs for a limited number of times or frequencies (temporal DOFs). Data taken in this manner is consistent with sinusoidal excitation using equal or unequal frequency spacing. The conceptual representation of several columns of this potential measurement matrix is shown in Figure 6.

Measurement data spaces involving many planes of measured data are the best possible modal identification situations since the data subspace includes contributions from temporal and spatial characteristics. This allows the best possibility of estimating the important contributing modal parameters.

It should be obvious that the data which defines the subspace needs to be acquired in a consistent measurement process, in order for the algorithms to estimate accurate modal parameters. This fact has triggered the need to measure all of the data simultaneously and has led to past advancements in data acquisition, digital signal processing and instrumentation designed to facilitate this measurement problem.

### 3.4 Concept: Fundamental Modal Identification Models

Rather than using a physically based mathematical model, the common characteristics of different modal parameter estimation algorithms can be more readily identified by using a matrix polynomial model. One way of understanding the basis of this model can be developed from the polynomial model used historically for the frequency response function.

$$\frac{X_p(\omega)}{F_\alpha(\omega)} = \frac{\beta_m(j\omega)^m + \beta_{m-1}(j\omega)^{m-1} + \cdots + \beta_1(j\omega) + \beta_0(j\omega)^0}{\alpha_m(j\omega)^m + \alpha_{m-1}(j\omega)^{m-1} + \cdots + \alpha_1(j\omega) + \alpha_0(j\omega)^0}$$

This can be rewritten:

$$\frac{X_p(\omega)}{F_\alpha(\omega)} = \sum_{k=0}^{n} \frac{\beta_k(j\omega)^k}{\sum_{k=0}^{n} \alpha_k(j\omega)^k}$$

Further rearranging yields the following equation that is linear in the unknown $\alpha$ and $\beta$ terms:
This model can be generalized to represent the general multiple input, multiple output case as follows:

\[
\sum_{k=0}^{n} \alpha_k \{ X(\omega) \} = \sum_{k=0}^{n} \beta_k \{ F(\omega) \},
\]

where

\[
\{ H(\omega) \} = \begin{bmatrix}
H_{10}(\omega) & 0 \\
H_{20}(\omega) & 0 \\
\vdots & \vdots \\
H_{m0}(\omega) & 0 \\
\end{bmatrix} \text{ and } \{ R \} = \begin{bmatrix}
\cdots \\
1 \\
\cdots \\
0 \\
\end{bmatrix}
\]

The above model, in the frequency domain, corresponds to an AutoRegressive-Moving-Average (ARMA) model that is developed from a set of finite difference equations in the time domain. The general matrix polynomial concept recognizes that both the time and frequency domain models generate essentially the same matrix polynomial models. For that reason, the Unified Matrix Polynomial Approach (UMPA) terminology is used to describe both domains since the ARMA terminology has been connected primarily with the time domain. Equation (15) can be rearranged into a linear matrix equation as shown in Appendices A, E and F.

Paralleling the development of Equation (13), a time domain model representing the relationship between a single response degree of freedom and a single input degree of freedom can be stated as follows:

\[
\sum_{k=0}^{n} \alpha_k x(t_i, \omega) = \sum_{k=0}^{n} \beta_k f(t_i, \omega)
\]

In the time domain, this model is commonly known as the AutoRegressive Moving-Average (ARMA(m,n)) model. For the general multiple input, multiple output case:

\[
\sum_{k=0}^{n} \{ \alpha_k \} \{ x(t_i, \omega) \} = \sum_{k=0}^{n} \{ \beta_k \} \{ f(t_i, \omega) \}
\]

If the discussion is limited to the use of free decay or impulse response function data, the previous time domain equations can be simplified by noting that the forcing function can be assumed to be zero for all time greater than zero. If this is the case, the \{ \beta_k \} coefficients can be eliminated from the equations.

\[
\sum_{k=0}^{n} \{ \alpha_k \} \left\{ h_{ij}(t_i, \omega) \right\} = 0
\]

Equation (18) can be rearranged into a linear matrix equation as shown in Appendices A, B and C.

In light of the above discussion, it is now apparent that most of the modal parameter estimation processes available could have been developed by starting from a general matrix polynomial formulation that is justifiable based upon the underlying matrix differential equation. The general matrix polynomial formulation yields essentially the same characteristic matrix polynomial equation, for both time and frequency domain data. For the frequency domain data case, this yields:

\[
\{ \alpha_m \} s^m + [\alpha_{m-1}] s^{m-1} + [\alpha_{m-2}] s^{m-2} + \ldots + [\alpha_0] = 0
\]

For the time domain data case, this yields:

\[
[\alpha_m] z^m + [\alpha_{m-1}] z^{m-1} + [\alpha_{m-2}] z^{m-2} + \ldots + [\alpha_0] = 0
\]

With respect to the previous discussion of model order, the characteristic matrix polynomial equation, Equation (19) or Equation (20) has a model order of \( m \) and the number of modal frequencies or roots that will be found from this characteristic matrix polynomial equation will be \( m \) times the size of the coefficient matrices \{ \alpha \}. In terms of sampled data, the time domain matrix polynomial results from a set of finite difference equations and the frequency domain matrix polynomial results from a set of linear equations where each equation is formulated at one of the frequencies of the measured data. This distinction is important to note since the roots of the matrix characteristic equation formulated in the time domain are in the \( z \)-domain (\( z \)) and must be converted to the frequency domain (\( \lambda \)) while the roots of the matrix characteristic equation formulated in the frequency domain (\( \lambda \)) are already in the desired domain \( \omega \). Note that the roots that are estimated in the time domain are limited to maximum values determined by the Nyquist Sampling Theorem relationship (discrete time steps).
TABLE 2. Summary of Modal Parameter Estimation Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Domain</th>
<th>Matrix Polynomial Order</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time</td>
<td>Freq</td>
<td>Zero</td>
</tr>
<tr>
<td>Complex Exponential Algorithm (CEA)</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Least Squares Complex Exponential (LSCE)</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Polynreference Time Domain (PTD)</td>
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<td>•</td>
</tr>
<tr>
<td>Ibrahim Time Domain (ITD)</td>
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<td>•</td>
</tr>
<tr>
<td>Multi-Reference Ibrahim Time Domain (MRITD)</td>
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<td>•</td>
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<tr>
<td>Eigensystem Realization Algorithm (ERA)</td>
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<tr>
<td>Polynreference Frequency Domain (PFD)</td>
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<td>Complex Mode Indication Function (CMIF)</td>
<td>•</td>
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</tr>
</tbody>
</table>

Using this general formulation, the most commonly used modal identification methods can be summarized as shown in Table 2.

The high order model is typically used for those cases where the system is undersampled in the spatial domain. For example, the limiting case is when only one measurement is made on the structure. For this case, the left hand side of the general linear equation corresponds to a scalar polynomial equation with the order equal to or greater than the number of desired modal frequencies. This type of high order model may yield significant numerical problems for the frequency domain case.

The low order model is used for those cases where the spatial information is complete. In other words, the number of independent physical coordinates is greater than the number of desired modal frequencies. For this case, the order of the left hand side of the general linear equation, Equation (15) or (18), is equal to one or two.

The zero order model corresponds to a case where the temporal information is neglected and only the spatial information is used. These methods directly estimate the eigenvectors as a first step. In general, these methods are programmed to process data at a single temporal condition or variable. In this case, the method is essentially equivalent to the single-degree-of-freedom (SDOF) methods which have been used with frequency response functions. In other words, the zeroth order matrix polynomial model compared to the higher order matrix polynomial models is similar to the comparison between the SDOF and MDOF methods used historically in modal parameter estimation.

3.5 Concept: Two Stage Linear Solution Procedure

Almost all modal parameter estimation algorithms in use at this time involve a two stage linear solution approach. For example, with respect to Equations (7) through (10), if all modal frequencies and modal participation vectors can be found, the estimation of the complex residues can proceed in a linear fashion. This procedure of separating the nonlinear problem into a multi-stage linear problem is a common technique for most estimation methods today.

For the case of structural dynamics, the common technique is to estimate modal frequencies and modal participation vectors in a first stage and then to estimate the modal coefficients plus any residuals in a second stage.

Therefore, based upon Equations (11) through (20), most modern modal identification algorithms can be outlined as follows:

- Load Measured Data into Linear Equation Form (Equation (15) or (18)).
  - Limit Temporal Information (Filter)
  - Limit Spatial Information (Siefe)
  - Limit Temporal Information (Decimate)
  - Perform Coefficient Condensation (Low Order Models)
  - Perform Equation Condensation
- Find Scalar or Matrix Autoregressive Coefficients (\(a_k\)).
- Solve Matrix Polynomial for Modal Frequencies.
  - Formulate Companion Matrix
  - Obtain Eigenvalues of Companion Matrix (\(\lambda\) or \(z\)).
  - Convert Eigenvalues from \(z\) to \(\lambda\), (time domain only).
  - Obtain Modal Participation Vectors \(l, p\) or Modal Vectors \(w, p\), from Eigenvectors of the Companion Matrix.
- Find Residues and Modal Scaling from Equations (7) through (10).

In order to estimate complex conjugate pairs of roots, at least two equations must be used from each block of data in the data space. This means that at least two sets of time data must be utilized from every impulse response function or the positive and negative frequency information for each frequency utilized from the frequency response function data. In order to develop enough equations to solve for the unknown matrix coefficients, further information is
taken from the same block of data or from other blocks of data in the data space until the number of equations equals or exceeds the number of unknowns (coefficients). The exact form and requirements for Equation (15) or (18) for many of the commonly used modal identification algorithms is given in Appendices A through F.

Once the matrix coefficients ([a]) have been found, the modal frequencies (ω, or ζ), can be found using a number of numerical techniques. While in certain numerical situations, other numerical approaches may be more robust, a companion matrix approach yields a consistent concept for understanding the process. Therefore, the roots of the characteristic equation can be found as the eigenvalues of the associated companion matrix. The companion matrix can be formulated in one of several ways. The most common formulation is as follows:

\[
\begin{bmatrix}
-\alpha_m + \frac{1}{\beta_m} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

(21)

Note again that the numerical characteristics of the eigenvalue solution of the companion matrix will be different for low order cases compared to high order cases for a given data set. The companion matrix can be used in the following eigenvalue formulation to determine the modal frequencies for the original matrix coefficient equation:

\[
[C] \{X\} = \lambda \{I\} \{X\} \tag{22}
\]

The eigenvectors that can be found from the eigenvalue-eigenvector solution utilizing the companion matrix may, or may not, be useful in terms of modal parameters. The eigenvector that is found, associated with each eigenvalue, is of length model order times matrix coefficient size. In fact, the unique (meaningful) portion of the eigenvector is of length equal to the size of the coefficient matrices and is repeated in the eigenvector a model order number of times. Each unique portion of the eigenvector is repeated, the unique portion is multiplied by a scalar multiple of the associated modal frequency. Therefore, the eigenvectors of the companion matrix have the following form:

\[
\{\lambda^\ast\{\psi\}\}, \quad \ldots, \quad \{\lambda_n^\ast\{\psi\}\}, \quad \ldots
\]

(23)

Note that, unless the size of the coefficient matrices is at least as large as the number of measurement degrees of freedom, only a partial set of modal coefficients, the modal participation coefficients (Lv), will be found. For the case involving scalar coefficients, no meaningful modal coefficients will be found.

If the size of the coefficient matrices, and therefore the modal participation vector, is less than the largest spatial dimension of the problem, then the modal vectors are typically found in a separate solution process using one of Equations (7) through (10). Even if the complete modal vector (ψr) of the system is found from the eigenvectors of the companion matrix approach, the modal scaling and modal participation vectors for each modal frequency are normally found in this second stage formulation.

Once the modal frequencies and modal participation vectors (or modal vectors) have been estimated, the associated modal vectors (or modal participation vectors and modal scaling (residues) can be found with standard least squares methods in either the time or frequency domain using one of the Equations (7) through (10). The most common approach is to estimate residues in the frequency domain utilizing residuals if appropriate.

The residues are calculated from the product of the modal participation vectors L and the modal coefficients (Lv) (Lvp = Lvψr). Note that if one column q of the modal participation matrix L is normalized to unity, the modal coefficients that are found will be equal to the residues for that reference Lvpq.

### 3.6 Concept: Data Filtering/Sieving/Decimation

For almost all cases of modal identification, a large amount of redundancy or overdetermination exists. This means that the number of equations available compared to the number required to form an exactly determined solution, defined as the overdetermination factor, will be quite large. Beyond some value of overdetermination factor, the additional equations contribute little to the result but may add significantly to the solution time. For this reason, the data space is often filtered (limited within minimum and maximum temporal axis values), sieved (limited to prescribed input DOFs and/or output DOFs) and/or decimated (limited number of equations from the allowable temporal data) in order to obtain a reasonable result in the minimum time. For frequency response function (frequency domain) data, the filtering process normally involves limiting the usable data within a minimum and maximum frequency according to the desired frequency range of interest. For impulse response function (time domain) data, the filtering process normally involves limiting the data within a starting and ending time value. Data sieving involves limiting the data set to certain degrees of freedom that are of primary interest. This normally involves restricting the data to specific directions (X, Y and/or Z directions) or specific locations or groups of degrees of freedom, such as components of a large structural system. Even when the available data has been filtered and sieved, as described above, a large overdetermination factor may still exist. When this occurs, the number of equations and, therefore, the overdetermination factor can be further reduced by decimating some of the remaining available equations. For frequency domain data, this means that information at selected frequencies within the frequency range of
interest can be eliminated. For time domain data, this means that information from different portions of the remaining available time range (overlaps, number of overlaps, time shift between overlaps) can be eliminated.

3.7 Concept: Coefficient Condensation

For the low order modal identification algorithms, the number of physical coordinates (typically $N_p$) is often much larger than the number of desired modal frequencies ($2N$). For this situation, the numerical solution procedure is constrained to solve for $N_p$, or $2N$, modal frequencies. This can be very time consuming and is unnecessary. The number of physical coordinates ($N_p$) can be reduced to a more reasonable size ($N_p \to N_c$ or $N_p \to 2N_c$) by using a decomposition transformation from physical coordinates ($N_p$) to the approximate number of effective modal frequencies ($N_c$). Currently, singular value decompositions (SVD) or eigenvalue decompositions (ED) are used to preserve the principal modal information prior to formulating the linear equation solution for unknown matrix coefficients $[7,11]$. In most cases, even when the spatial information must be condensed, it is necessary to use a model order greater than two to compensate for distortion errors or noise in the data and to compensate for the case where the location of the transducers is not sufficient to totally define the structure.

3.8 Concept: Equation Condensation

Several important concepts should be delineated in the area of equation condensation methods. Equation condensation methods are used to reduce the number of equations based upon measured data to more closely match the number of unknowns in the modal parameter estimation algorithms. There are a large number of condensation algorithms available. Based upon the modal parameter estimation algorithms in use today, the three types of algorithms most often used are:

- Least Squares: Least squares (LS), weighted least squares (WLS), total least squares (TLS) or double least squares (DLS) is used to minimize the squared error between the measured data and the estimation model. Historically, this is one of the most popular procedures for finding a pseudo-inverse solution to an over specified system. The main advantage of this method is computational speed and ease of implementation while the major disadvantage is numerical precision.

- Transformations: There are a large number of transformation that can be used to reduce the data. In the transformation methods, the measured data is reduced by approximating the data by the superposition of a set of significant vectors. The number of significant vectors is equal to the amount of independent measured data. This set of vectors is used to approximate the measured data and used as input to the parameter estimation procedures.

- Singular Value Decomposition: Singular Value Decomposition (SVD) is an example of one of the more popular transformation methods. The major advantage of these methods is numerical precision and the disadvantage is computational speed and memory requirements.

- Coherent Averaging: Coherent averaging is another popular method for reducing the data. In the coherent averaging method, the data is weighted by performing a dot product between the data and a weighting vector (spatial filter). Information in the data which is not coherent with the weighting vectors is averaged out of the data. The method is often referred to as a spatial or modal filtering procedure. This method has both speed and precision but, in order to achieve precision, requires a good set of weighting vectors. In general, the optimum weighting vectors are connected with the solution which is unknown. It should be noted that least squares is an example of a non-coherent averaging process.

The least squares and the transformation procedures tend to weight those modes of vibration which are well excited. This can be a problem when trying to extract modes which are not well excited. The solution is to use a weighting function for condensation which tends to enhance the mode of interest. This can be accomplished in a number of ways:

- In the time domain, a spatial filter or a coherent averaging process can be used to filter the response to enhance a particular mode or set of modes. For example, by averaging the data from two symmetric exciter locations the symmetric modes of vibration can be enhanced. A second example would be using only the data in a local area of the system to enhance local modes. The third method is using estimates of the modes shapes as weighting functions to enhance particular modes.

- In the frequency domain, the data can be enhanced in the same manner as the time domain plus the data can be additionally enhanced by weighting the data in a frequency band near the natural frequency of the mode of interest.

Obviously, the type of equation condensation method that is utilized in a modal identification algorithm has a significant influence on the results of the parameter estimation process.

3.9 Concept: Model Order Determination

Much of the work concerned with modal parameter estimation since 1975 has involved methodology for determining the correct model order for the modal parameter model. Technically, model order refers to the highest power in the matrix polynomial equation. The number of modal frequencies found will be equal to the model order times the size of the matrix coefficients, normally $N_c$ or $N_p$. For a given algorithm, the size of the matrix coefficients is normally fixed; therefore, determining the model order is directly linked to estimating $N_c$, the number of modal frequencies that are of interest in the measured data. As has always been the case, an estimate for the minimum number of modal frequencies can be easily found by counting the number of peaks in the frequency response function in the frequency band of analysis. This is a minimum estimate of $N_c$ since the frequency response function measurement may be at a node of one or more modes of the system, repeated roots may exist and/or the frequency resolution of the measurement may be too coarse to observe modes that are closely spaced in frequency. For these reasons, an appropriate estimate of the order of the model is of prime concern and is the single most
important problem in modal parameter estimation.

In order to determine a reasonable estimate of the model order for a set of representative data, a number of techniques have been developed as guides or aids to the user. Much of the user interaction involved in modal parameter estimation involves the use of these tools. Most of the techniques that have been developed allow the user to establish a maximum model order to be evaluated (in many cases, this is set by the memory limits of the computer algorithm). Data is acquired based upon an assumption that the model order is equal to this maximum. In a sequential fashion, this data is evaluated to determine if a model order less than the maximum will describe the data sufficiently. This is the point that the user’s judgement and the use of various evaluation aids becomes important. Some of the commonly used techniques are:

- Error Chart
- Stability Diagram
- Mode Indication Functions
- Rank Estimation

3.9.1 Error Chart One method that has been used to directly indicate the correct model order is the error chart. Essentially, the error chart is a plot of the error in the model as a function of increasing model order. The error in the model is a normalized quantity that represents the ability of the model to predict data that was not involved in the estimate of the model parameters. When the model order is insufficient, this model error will be large but when the model error reaches the "correct" value, further increase in the model order will not result in a further decrease in the error. Figure 7 is an example of an error chart.

![Error Chart](image)

**Figure 7. Model Order Determination: Error Chart**

3.9.2 Stability Diagram A further enhancement of the error chart is the stability diagram. The stability diagram is developed in the same fashion as the error chart and involves tracking the estimates of frequency, damping, and possibly modal participation factors as a function of model order. As the model order is increased, more and more modal frequencies are estimated but, hopefully, the estimates of the physical modal parameters will stabilize as the correct model order is found. For modes that are very active in the measured data, the modal parameters will stabilize at a very low model order. For modes that were poorly excited in the measured data, the modal parameters may not stabilize until a very high model order is chosen. Nevertheless, the nonphysical (computational) modes will not stabilize during this process and can be sorted out of the modal parameter data set more easily. Note that inconsistencies (frequency shifts, leakage errors, etc.) in the measured data set will obscure the stability and render the stability diagram difficult to use. Normally, a tolerance, in percentage, is given for the stability of each of the modal parameters that are being evaluated. Figure 8 is an example of a stability diagram. In Figure 8, a summation of the frequency response function power is plotted on the stability diagram for reference.

![Stability Diagram](image)

**Figure 8. Model Order Determination: Stability Diagram**

3.9.3 Mode Indication Function Mode indication functions (MIF) are normally real-valued, frequency domain functions that exhibit local minima or maxima at the natural frequencies of real normal modes. One mode indication function can be plotted for each reference available in the measured data. The primary mode indication function will exhibit a local minima or maxima at each of the natural frequencies of the system under test. The secondary mode indication function will exhibit a local minimum at repeated or pseudo-repeated roots of order two or more. Further mode indication functions yield local minima or maxima for successively higher orders of repeated or pseudo-repeated roots of the system under test. The resulting plot of a multivariate mode indication function for a multiple reference case can be seen in Figure 9.

![Mode Indication Function](image)

**Figure 9. Multivariate Mode Indication Function**

The resulting plot of a complex mode indication function for a multiple reference case can be seen in Figure 10.
3.9.4 Rank Estimation A more recent model order evaluation technique involves the estimate of the rank of the matrix of measured data. An estimate of the rank of the matrix of measured data gives a good estimate of the model order of the system. Essentially, the rank is an indicator of the number of independent characteristics contributing to the data. While the rank cannot be calculated in an absolute sense, the rank can be estimated from the singular value decomposition (SVD) of the matrix of measured data. For each mode of the system, one singular value should be found by the SVD procedure. The SVD procedure finds the largest singular value first and then successively finds the next largest. The magnitude of the singular values are used in one of two different procedures to estimate the rank. The concept that is used is that the singular values should go to zero when the rank of the matrix is exceeded. For theoretical data, this will happen exactly. For measured data, due to random errors and small inconsistencies in the data, the singular values will not become zero but will be very small. Therefore, the rate of change of the singular values is used as an indicator rather than the absolute values. In one approach, each singular value is divided by the first (largest) to form a normalized ratio. This normalized ratio is treated much like the error chart and the appropriate rank (model order) is chosen when the normalized ratio approaches an asymptote. In another similar approach, each singular value is divided by the previous singular value forming a normalized ratio that will be approximately equal to one if the successive singular values are not changing magnitude. When a rapid decrease in the magnitude of the singular value occurs, the ratio of successive singular values drops (or peaks if the inverse of the ratio is plotted) as an indicator of rank (model order) of the system. Figure 11 is an example of these rank estimation procedures.

4. Summary - Future Trends

Modal parameter estimation is probably one of the most misunderstood aspects of the total experimental modal analysis process. It should be obvious that the modal identification algorithm cannot compensate for basic errors made in the measurement of the data used in the estimation process. Since most modal parameter estimation methods are mathematically intimidating, many users do not fully understand the ramifications of the decisions made during the measurement stages as well as later in the modal parameter estimation process. Ideally, by consolidating the conceptual approach and unifying the theoretical development of modal identification algorithms, increased understanding, with respect to general advantages and disadvantages of different algorithms, can be achieved. This sort of overview of modal parameter estimation can be used simply as a guide toward further study and understanding of the details of the individual modal identification algorithms.

5. Bibliography


Appendix A: High Order Time Domain Algorithms

Typical Algorithms
- Complex Exponential (CE)
- Least Squares Complex Exponential (LSCE)
- Polyreference Time Domain (PTD)

General Linear Equation Formulation
- High Order \((m \geq 2N/N_x \text{ or } mN_x \geq 2N)\)
- Matrix Coefficients \((N_x \times N_x)\)

Basic Equation:
\[
\begin{bmatrix}
    [h(t_0)]
    [h(t_1)]
    ... \\
    [h(t_m-1)]
\end{bmatrix}
\begin{bmatrix}
    \alpha_0 \\
    \alpha_1 \\
    ... \\
    \alpha_{m-1}
\end{bmatrix}
= -[h(t_m)]
\]
\hspace{1cm} (24)

where:
\[
[h(t_k)]_{N_x, N_x} = 
\begin{bmatrix}
    h_{k1}(t_{i+k}) & h_{k2}(t_{i+k}) & h_{k3}(t_{i+k}) & ... & h_{kp}(t_{i+k})
\end{bmatrix}
\]

Appendix B: First Order Time Domain Algorithms

Typical Algorithms
- Ibrahim Time Domain (ITD)
- Multiple Reference Time Domain (MRITD)
- Eigensystem Realization Algorithm (ERA)

General Linear Equation Formulation
- Low Order \((m = 1 \text{ and } 2N_x = 2N)\)
- Matrix Coefficients \((2N_x \times 2N_x)\)

Basic Equation:
\[
\begin{bmatrix}
    [h(t_0)]
    [h(t_1)]
\end{bmatrix}
\begin{bmatrix}
    \alpha_0
\end{bmatrix}
= -[h(t_2)]
\]
\hspace{1cm} (25)

where:
\[
[h(t_k)]_{N_x, 1} = 
\begin{bmatrix}
    h_{11}(t_{i+k}) & h_{12}(t_{i+k}) & h_{13}(t_{i+k}) & ... & h_{1p}(t_{i+k})
\end{bmatrix}
\]

Appendix C: Second Order Time Domain Algorithms

Typical Algorithms
- Time Domain Equivalent to Polyreference Frequency Domain

General Linear Equation Formulation
- Low Order \((m = 2 \text{ and } 2N_x = 2N)\)
- Matrix Coefficients \((N_x \times N_x)\)

Basic Equation:
\[
\begin{bmatrix}
    [h(t_0)]
    [h(t_1)]
\end{bmatrix}
\begin{bmatrix}
    \alpha_0
\end{bmatrix}
= -[h(t_2)]
\]
\hspace{1cm} (26)

where:
\[
[h(t_k)]_{N_x, N_x} = 
\begin{bmatrix}
    h_{k1}(t_{i+k}) & h_{k2}(t_{i+k}) & h_{k3}(t_{i+k}) & ... & h_{kp}(t_{i+k})
\end{bmatrix}
\]
Appendix D: High Order Frequency Domain Algorithms

Typical Algorithms
- Rational Fraction Polynomial (RFP)
- Orthogonal Polynomial (OP)

General Linear Equation Formulation
- High Order \((m \geq 2N/Ns\) or \(mN, \geq 2N\))
- Matrix Coefficients \((N, \times N,\)

Basic Equation:
\[
\begin{bmatrix}
[\alpha_{0}]
[\alpha_{1}]
\vdots
[\alpha_{n-1}]
[\beta_{0}]
\vdots
[\beta_{s}]
\end{bmatrix}
= - (j\omega_{0})^{n}[H(\omega_{0})]
\]
(27)

where:
\[
[H] = \begin{bmatrix}
H(\omega_{0}) & H(\omega_{0}) & \cdots & H(\omega_{0})
\end{bmatrix}
\]
\[
[R] = \begin{bmatrix}
R & -j\omega_{0} & \cdots & -j\omega_{0}^{n-1}
\end{bmatrix}
\]

Appendix E: First Order Frequency Domain Algorithms

Typical Algorithms
- Simultaneous Frequency Domain (SFD)
- Multi-Reference Frequency Domain (MRFD)
- Frequency Domain Equivalent to ITD, MRITD, ERA

General Linear Equation Formulation
- Low Order \((m = 1\) and \(2N, \geq 2N\))
- Matrix Coefficients \((2N, \times 2N,\)

Basic Equation:
\[
\begin{bmatrix}
H(\omega_{0}) & (j\omega_{0}) & H(\omega_{0}) & \cdots & (j\omega_{0})^{n-1} & H(\omega_{0})
\end{bmatrix}
\]
\[
[R] = \begin{bmatrix}
R & -j\omega_{0} & \cdots & -j\omega_{0}^{n-1}
\end{bmatrix}
\]
(28)

where:
\[
[H(\omega_{0})]_{1, N_{s}} = \begin{bmatrix}
H_{1}(\omega_{0}) & H_{2}(\omega_{0}) & \cdots & H_{p}(\omega_{0})
\end{bmatrix}
\]
\[
[R]_{1, N_{s}} = \begin{bmatrix}
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

Appendix F: Second Order Frequency Domain Algorithms

Typical Algorithms
- Polyreference Frequency Domain (PFD)

General Linear Equation Formulation
- Low Order \((m = 2\) and \(2N, = 2N\))
- Matrix Coefficients \((N, \times N,\)

Basic Equation:
\[
\begin{bmatrix}
H(\omega_{0}) & (j\omega_{0}) & H(\omega_{0}) & \cdots & (j\omega_{0})^{n-1} & H(\omega_{0})
\end{bmatrix}
\]
\[
[R] = \begin{bmatrix}
R & -j\omega_{0} & \cdots & -j\omega_{0}^{n-1}
\end{bmatrix}
\]
(29)

where:
\[
[H(\omega_{0})]_{1, N_{s}} = \begin{bmatrix}
H_{1}(\omega_{0}) & H_{2}(\omega_{0}) & \cdots & H_{p}(\omega_{0})
\end{bmatrix}
\]
\[
[R]_{1, N_{s}} = \begin{bmatrix}
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]