ABSTRACT

In this paper the discrete - continuous model of a multi-storeyed building is considered. It is assumed that columns are deformed only transversely, whereas ceilings and foundations are undeformable. The columns are assumed to contain microcracks which are taken into account by means of the equivalent shear modulus. The method of dynamical analysis is based on the theory of propagation of one dimensional shear elastic waves. Detailed discussion is presented for a three storey building with a main floor jump-like change of the porosity coefficient. Final equations for N storey building are given and some numerical results are presented for a three storey structure.

NOMENCLATURE

- $x$: vertical coordinate
- $t$: time
- $l$: height of column
- $y_i$: displacement of the $i$th column
- $\dot{y}_i$: acceleration
- $\rho_i$: density
- $c_i$: shear wave speed
- $f_i$: left shear wave
- $g_i$: right shear wave
- $D_i$: coefficient of viscous damping
- $G_i$: shear modulus
- $A$: column cross-section
- $\nu$: Young modulus
- $\nu_i$: Poisson’s ratio
- $\phi$: porosity coefficient
- $Y_0$: external loading

1. INTRODUCTION

Studies of the vibration and the wave propagation characteristics of discrete systems have long been of interest to researchers in the fields of geophysics, acoustics and structures. More recently further advances in the study of elastic waves in discrete media have been necessary for the analysis of the dynamic behavior of composite materials. Common to all of these studies are the interactions among the components, which manifest themselves in the form of reflection and transmission agents and give rise to geometric dispersion. These interactions depend among other factors, upon mechanical properties, direction of propagation and frequency of the incident waves, and number and nature of the interfacial conditions.

Discrete systems are encountered in practically all structural applications and range from the simple spring-mass systems to the much more complicated structures consisting of multilayered anisotropic media. The literature on the analysis of waves in layered systems concerns mainly the determination of dispersion relations in undamped semi-infinite systems neglecting discrete masses. This approach has a long history and attracted considerable attention in the past as well as in recent years. To mark only some of the previous contribution in this field one should mention [1] and [2], where a matrix formulation for computing transmission coefficients in a layered medium was developed (so called Huckle - Thompson method). This approach was later extended in [3] to study transient incident P waves and in [4] to include damping in soil layers. A variety of research papers in this area was published more recently [5-10] with the proposed models being one and multi-dimensional. Although multi-dimensional models should more precisely describe physical phenomena in real systems, efficient methods for their investigations have not been worked out as yet.
It should be pointed out that the models of engineering constructions and mechanical systems ought to consist quite often not only of elements having various mechanical properties but also of discrete masses representing elements, the deformations of which can be neglected. There exists a vast literature (to mention only [11] and [12]) dealing with this type of engineering structures with both discrete and continuous models taken into account, including a jump like change of physical properties at the base of the system. In the present paper an attempt is made to consider discrete - continuous models of engineering constructions which exhibit microcracks over the entire system. For this reason, the model representing a simplified structure has been adopted which consists of straight vertically rigid but horizontally deformable columns rigidly connected to the undeformable foundation and floors. These columns are assumed to contain microcracks which are taken into account by means of the equivalent shear modulus. In order to describe the dynamical behaviour of this model of a multi-storeyed building, the theory of propagation of one-dimensional shear elastic waves is employed together with appropriate initial and boundary conditions. Internal and external damping of the system is described by an equivalent damping. This approach leads to a system of linear ordinary differential equations of retarded type with constant coefficients.

It should be noted that the dynamical analysis of shear deformations in multi-storeyed buildings based on similar analytical approach is presented in [13] and [14] for homogeneous structures - all columns are made of the same material where in [14] the influence of variable length of columns is investigated. In the present paper, this method is extended to the case of n-storey building with microcracks in the columns and the influence of microcracks on the system under dynamic conditions is investigated. Some numerical results are presented in graphical form for a three storey building for the cases of uniform equivalent shear modulus throughout the structure, and jump like change in the distribution of the equivalent shear modulus between the main floor and the rest of the building - the so called base isolated system.

2. PHYSICAL MODEL

Consider a multi-storeyed building having columns rigidly connected to the foundations and floors. It is assumed that the foundation and floors are undeformable and are displaced by plane motion during the time of seismic loading. The masses of the foundation and ceilings are denoted respectively by: \( m_0 \) - mass of the foundation, \( m_i \) - masses of the ceilings, \( i = 1, 2, ..., N \).

It is assumed that all points of the foundation of the building have the same horizontal accelerations as the "surroundings" of the foundation. A graph of these accelerations is given as an example in Fig. 1 [15]. From Fig. 1 it follows that the acceleration curve \( \ddot{y}_g \) is irregular. Therefore in the calculations it is approximated suitably by segments of straight lines as shown in Figs. 2 or 3. This approximation is carried out depending on the oscillation frequency of the horizontal acceleration given in Fig. 1. When the period of oscillation is larger or slightly different from the fundamental period of free vibrations of the building, the horizontal accelerations (Fig. 1) can then be approximated by the following function

\[
\ddot{y}(t) = \sum_{k=0}^{m} H(t-t_k) [a_k + b_k(t-t_k)].
\]

However, when the period is much smaller than the fundamental period of free vibrations of the building it is assumed that the graph of horizontal acceleration is of the Dirac type

\[
\ddot{y}(t) = \sum_{k=0}^{m} a_k \delta(t-t_k).
\]

In both cases of kinematic forces different kinds of waves will occur, and thereby different stresses and displacements of the column cross-sections.

It is assumed that the vertical columns between the foundation and first ceiling and between the ceilings of the building, subject to seismic loading, have the same shapes as regards to elastic strains, i.e. the velocities, strains and displacements for the given height of a building are the same [15]. It is assumed that the column cross-sections during seismic loading remain plane cross-sections parallel to the cross-sections of the building. These columns are characterized by the following parameters: \( G \) - equivalent shear modulus of elasticity, \( A \) - column cross-sectional area, \( \rho \) - density, \( \ell \) - height of the column.

With the above assumption, the physical model of a multi-storey building with continuously distributed parameters [16] which is shown in Fig. 4 is adopted for consideration. In this model the equivalent damping in selected cross-sections of the columns is considered also. Damping forces, which load the foundation with mass \( q \) and the ceilings with masses \( q = 1, 2, ..., N \), are assumed in the form

\[
P_D(x,t) = -D_x \frac{\partial y_i(x,t)}{\partial t} \quad \text{for } x = il,
\]

\[
i = 0, ..., N - 1
\]

\[
P_D(x,t) = -D_N \frac{\partial y_N(x,t)}{\partial t} \quad \text{for } x = Nl,
\]

where \( y_i(x,t) \) is the transverse displacement of the ith column, and \( D_x, i = 0, 1, ..., N \), are the equivalent damping coefficients of the viscous type. The damping forces \( P_{D_D}(i) \) of selected column cross-sections take into consideration internal damping (continuously distributed) and external damping. Therefore it is assumed that the external damping occurring between the elements which are
displaced in relative motion during seismic loading is also of the viscous type just as the internal damping.

In the literature there are many papers concerning effective methods which take into account microcracks in a real material. For example, in papers [17-21], relations for equivalent elastic moduli depending on the material constants of an elastic medium and the number of microcracks are obtained. The assumptions are that the microcracks have a statistically uniform distribution in the material and do not change its homogeneous and isotropic elastic properties, that the volume of all the microcracks is small compared with the total volume, and that the total volume contains a large number of microcracks. Then a volume element with microcracks can be regarded as a homogeneous isotropic continuum with equivalent elastic constants.

Estimates of the overall elastic moduli of an elastic solid which contains microcracks are of considerable theoretical and practical interest. Different methods have been developed by various authors. A survey of approaches to obtain equivalent elastic constants is given in [19], where also relations for equivalent elastic constants obtained by various authors are compared.

In the present paper the relation for the equivalent shear modulus $G$ is taken from [17,18] in the form

$$G = G_0(1 - \phi) \left(1 - \frac{6K_0 + 12G_0}{9K_0 + 8G_0} \phi \right),$$

where $\phi[0,1]$ is the porosity coefficient, $K_0 = E/(3(1-2\nu))$, $G_0 = E/(2(1+\nu))$, $E$ is Young's modulus, and $\nu$ is Poisson's ratio. In Fig. 5 several diagrams of $G/G_0$ are plotted as functions of the porosity coefficient with $\nu = 0.3$ [18,19]. The diagram corresponding to (4) is marked by a continuous line, the results according to the papers of R. Hill [20] and B. Budiansky [21] are marked by the broken line, and the diagram of S. Nemat-Nasser and M. Taya is marked by the dash-dot line [19].

The aim of the present paper is to investigate the influence of microcracks on the behaviour of the considered system under dynamic conditions. (This influence is discussed through the change of the wave speed which depends on the porosity coefficient.)

3. GOVERNING EQUATIONS

In order to simplify presentation of the method of dynamical analysis, a three storey building is discussed first, with the physical properties of the ground floor columns being different than for the remaining floors. This represents, within the framework of adopted model, a base-isolated structure and $G$, $\rho_i$ for $i = 1,2,3$ are $G_i$ and $\rho_i$ for the ground floor and $G_0 = G$, $\rho_0 = \rho$ for the remaining columns. For the discrete-continuous model the investigation of velocities $\partial y_i(x,t)/\partial t$, strains $\partial \gamma_i(x,t)/\partial x$, and displacements $y_i(x,t)$ of the columns is reduced to the solution of 3 wave equations:

$$\frac{\partial^2 y_i(x,t)}{\partial t^2} - c_i^2 \frac{\partial^2 y_i(x,t)}{\partial x^2} = 0, \quad i = 1,2,3$$

where $c_i^2 = G_i/\rho_i$, with the following conditions:

$$AG_i \frac{\partial y_i(x,t)}{\partial x} - m_i \frac{\partial y_i(x,t)}{\partial t} - D_i \frac{\partial^2 y_i(x,t)}{\partial t^2} = 0, \quad x = 0$$

$$y_1(x,t) = y_2(x,t), \quad x = \ell$$

$$y_2(x,t) = y_3(x,t), \quad x = 2\ell$$

$$AG \frac{\partial y_2(x,t)}{\partial x} - AG \frac{\partial y_2(x,t)}{\partial x} - m_1 \frac{\partial^2 y_2(x,t)}{\partial t^2}$$

$$-D_1 \frac{\partial y_2(x,t)}{\partial t} = 0, \quad x = \ell$$

$$AG \frac{\partial y_3(x,t)}{\partial x} - AG \frac{\partial y_3(x,t)}{\partial x} - m_2 \frac{\partial^2 y_3(x,t)}{\partial t^2}$$

$$-D_2 \frac{\partial y_3(x,t)}{\partial t} = 0, \quad x = 2\ell$$

$$AG \frac{\partial^2 y_2(x,t)}{\partial x^2} - AG \frac{\partial^2 y_2(x,t)}{\partial x^2} + D_3 \frac{\partial^2 y_3(x,t)}{\partial t^2} = 0, \quad x = 3\ell$$

and initial conditions,

$$y_i(x,t) = \frac{\partial y_i(x,t)}{\partial t} = 0, \quad for \; t = 0, \; i = 1,2,3.$$ (7)

Upon the introduction of non-dimensional quantities:

$$\bar{x} = \frac{x}{\ell}, \quad \bar{t} = \frac{\ell}{T}, \quad \bar{y}_i = \frac{y_i(y_o)}{\bar{y}_o} = \frac{y_i}{c_i^2 \bar{y}_o}$$

$$K_i = \frac{A \bar{y}_i}{m_i}, \quad D_i = \frac{D_i^2}{c_i}, \quad \alpha_i = \frac{c_i}{c_1}, \quad B_i = \frac{G_i}{G_1}$$

for $i = 1,2,3$, with $\alpha_2 = \alpha_3 = \alpha$ and $B_2 = B_3 = B$. Now the relations (5-7) become:

$$\frac{\partial^2 \bar{y}_1}{\partial \bar{t}^2} - \alpha_1 \frac{\partial^2 \bar{y}_1}{\partial \bar{x}^2} = 0, \quad i = 1,2,3$$

$$K_0 \frac{\partial \bar{y}_1}{\partial \bar{x}} - \bar{Y}_o - K_0 D_0 \frac{\partial \bar{y}_1}{\partial \bar{t}} = 0, \quad x = 0$$

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\[ y_1 = y_2, \quad x = 1 \quad y_2 = y_3, \quad x = 2 \]

\[ K_1 \left[ B \frac{\partial y_1}{\partial x} - \frac{\partial y_1}{\partial t} \right] - \frac{\partial^2 y_1}{\partial t^2} - K_1 D_1 \frac{\partial y_2}{\partial t} = 0, \quad x = 1 \]

\[ K_2 \left[ B \frac{\partial y_2}{\partial x} - B \frac{\partial y_2}{\partial t} \right] - \frac{\partial^2 y_2}{\partial t^2} - K_2 D_2 \frac{\partial y_3}{\partial t} = 0, \quad x = 2 \]

\[ K_3 \frac{\partial y_3}{\partial x} + \frac{\partial^2 y_3}{\partial t^2} + K_3 D_3 \frac{\partial y_3}{\partial t} = 0, \quad x = 3 \]

and

\[ y_1 = \frac{\partial y_1}{\partial t} = 0, \quad t = 0, \quad i = 1, 2, 3. \quad (11) \]

For convenience all \( y_i(x,t) \) functions are written as \( y_i \) and all bars are omitted.

The solutions of equations (9) are sought in the nondimensional form

\[ y_i(x,t) = f_i(\alpha(t-\tau_0) - (x-x_0)) + g_i(\alpha(t-\tau_0) + (x-x_0)) \]

where the function \( f_i \) represents a wave propagating to the right and \( g_i \) represents a wave propagating to the left side of the \( i \)-th column as the result of application of the loading \( Y_0(t) \). It is assumed that the functions \( f_i \) and \( g_i \) are continuous and for negative arguments equal zero. By taking into account that the first perturbation in the first column starts at the instant \( \tau_0 = 0 \) at the bottom of that column, \( x_0 = 0 \), and the first perturbation in the \( i \)-th column starts at its bottom as well, one gets:

\[
\begin{align*}
  y_1(x,t) &= f_1(t-x) + g_1(t+x) \\
  y_2(x,t) &= f_2(\alpha(t-1) - x + 1) + g_2(\alpha(t-1) + x - 1) \\
  y_3(x,t) &= f_3(\alpha(t-1) - 1 + x + 2) + g_3(\alpha(t-1) - 1 + x - 2)
\end{align*}
\]

(12)

By substituting (12) into the boundary conditions (10) a system of equations is obtained for functions \( f_i \) and \( g_i \). One observes that there occurs simple relationship between arguments of the functions appearing in the same equation. Upon denoting the largest argument in each equation by the variable \( \zeta \), the arguments of the remaining functions are then shifted by a constant. This procedure leads to the following final system for six linear, ordinary differential equations of the first and second order for the unknown functions \( f_i(\zeta) \) and \( g_i(\zeta) \), \( i = 1, 2, 3 \):

\[
\begin{align*}
  g_1'(\zeta) &= -f_1'(\zeta - 2) + f_1'(\zeta - 2) + g_1'(\zeta - 2) \\
  g_2'(\zeta) &= -f_2'(\zeta - 2) + f_2'(\zeta - 2) + g_2'(\zeta - 2) \\
  g_3'(\zeta) &= r_3 g_3'(\zeta) - f_3'(\zeta - 2) + s_3 g_3'(\zeta - 2) \\
  f_1'(\zeta) &= g_1'(\zeta)(1 - D_o)/(1 + D_o) - \frac{r_1(\zeta)}{(K_o(1 + D_o))} \\
  f_2'(\zeta) &= r_2 f_2'(\zeta) - g_2'(\zeta) + s_2 g_2'(\zeta) + t_2 f_2'(\zeta) \\
  f_3'(\zeta) &= r_3 f_3'(\zeta) - g_3'(\zeta) + s_3 g_3'(\zeta) + t_3 f_3'(\zeta)
\end{align*}
\]

where

\[
\begin{align*}
  r_1 &= K_1 \frac{B}{\alpha} - D_1 + 1, & s_1 &= -K_1 \frac{B}{\alpha} - D_1 - 1, \\
  r_2 &= K_2 \frac{B}{\alpha} + D_2, & s_2 &= -K_2 D_2, \\
  r_3 &= K_3 \frac{B}{\alpha} + D_3, & s_3 &= K_3 \frac{B}{\alpha} - D_3, \\
  t_1 &= 2K_1, & t_2 &= 2K_2
\end{align*}
\]

The system of equations (13) should be solved in the given sequence in the successive intervals of the argument \( \zeta \). Because functions \( f_i \) and \( g_i \) equal zero for negative arguments, hence when solving equations (13) in this sequence the right hand sides of these equations are always known if loading \( Y_0(\zeta) \) is a known function of time.

A similar procedure to the one outlined above may be applied to the case of a multi-storey building, leading directly to the set of final equations. If, for example, the porosity coefficient is the same, but arbitrary, for all columns, then this leads to the set of 2\( N \) differential equations for the functions \( f_i(\zeta) \) and \( g_i(\zeta) \), \( i = 1, 2, \ldots, N \):

\[
\begin{align*}
  g'_i(\zeta) &= -f'_i(\zeta - 2) + f'_{i+1}(\zeta - 2) + g'_{i+1}(\zeta - 2), \\
  &\text{for} \quad i = 1, 2, \ldots, N - 1 \\
  g''_N(\zeta) &= r_N g'_N(\zeta) - f''_{N}(\zeta) + s_N g''_{N}(\zeta - 2) \\
  f'_i(\zeta) &= g'_i(\zeta)(1 - D_o)/(1 + D_o) - \frac{r_i(\zeta)}{(K_o(1 + D_o))} \\
  f''_i(\zeta) &= r_{i+1} f'_i(\zeta) - g''_i(\zeta) + s_{i+1} g''_i(\zeta) + t_{i+1} f''_i(\zeta), \\
  &\text{for} \quad i = 2, 3, \ldots, N
\end{align*}
\]

where

\[
\begin{align*}
  r_i &= K_i(2 + D_o), & s_i &= -K_i D_i, & t_i &= 2K_i, \\
  &\text{for} \quad i = 1, 2, \ldots, N - 1; \\
  r_N &= K_N(1 + D_o), & s_N &= K_N(1 - D_N)
\end{align*}
\]
4. NUMERICAL PROCEDURE AND EXAMPLE

One notes that all second order differential equations in the set (13) have the form
\[
f''(\zeta) + r f'(\zeta) = -g''(\zeta-1) + s g'(\zeta-1) + h(\zeta-k),
\]
where \( g'(\zeta-1) \) and \( h(\zeta-k) \) are given functions, and whose solution for \( \zeta \geq \zeta_0 \) is as follows
\[
f'(\zeta) = e^{-r(\zeta-\zeta_0)} \left[ e^{-s(\zeta-\zeta_0)} \left[ e^{-r(\zeta-\zeta_0)} \int_{\zeta_0}^{\zeta} \left( -g''(\xi-1) + s g'(\xi-1) + h(\xi-k) \right) d\xi \right] + f'(\zeta_0) \right].
\]

After integrating by parts this finally gives
\[
f'(\zeta) = f'(\zeta_0) e^{-r(\zeta-\zeta_0)} + \int_{\zeta_0}^{\zeta} \left[ e^{-r(\xi-\zeta_0)} \left( r g'(\xi-1) - s [g'(\xi-1)]_0^{\xi} + h(\xi-k) \right) d\xi \right].
\]

Since both integrals in equation (16) can be easily evaluated because \( g'(x-1) \) and \( h'(x-k) \) are known functions, one can apply this equation to transform all second order differential equations in (13) into first order equations. Any numerical integration procedure can be used, giving simple, stable and effective numerical procedure for solving equations (13).

As an example, a three-storey building is considered. For the purpose of numerical evaluation, the model is further simplified by the assumption that all non-dimensional coefficients pertaining to geometrical and physical properties are equal to one, and the external loading of the structure is of the Dirac type \( \eta_0 = \delta(t) \). The choice of coefficients and loading allows one to analyze the influence of nonhomogeneity on the structure, rather than the design itself, which involves a multitude of parameters.

In Figs. 6, 7 and 8, shear strains and velocities are shown for the cross-sections \( x = 0.75 \) and \( x = 2.25 \) for the external loading \( \eta_0 \) of the Dirac's delta type where the delta is approximated by a rectangle with the base \( \Delta x = 0.02 \) and the height \( 1/\Delta x \). These two cross-sections are chosen to show response at various floors of the building to the jump-like change in the wave speed between the main floor and the rest of the building. This type of structure is the one most extensively studied, with the base consisting often of laminated rubber with and without a lead core [11]. This system is used in a number of buildings in Europe, Japan and New Zealand.

In Fig. 6, shear strains are shown for the cross-section \( x = 0.75 \). Nondimensional shear wave speed in the columns of the ground floor is always equal one and is independent of the porosity coefficient, and equivalent shear modulus \( (c_3 - \beta_3 - 1) \) is decreasing rapidly with time due to geometry and damping of the system. In Figs. 7 and 8 strains \( \eta_0/\Delta x \) and velocities \( \eta_0/\beta \eta_0/\Delta x \) are shown for the cross-section \( x = 2.25 \) for the homogeneous case, \( c_1 = c_2 = c_3 = 1 \) (continuous line) and the nonhomogeneous case, when \( c_1 = 1 \) and \( c_2 = c_3 = 1/\sqrt{2} \) (dashed line). This nonhomogeneity is caused by the jump-like change in the distribution of the porosity coefficient between the main floor and the rest of the building. Assuming porosity coefficient \( \phi = 0 \) for the main floor columns and \( \phi = 0.3 \) for the remaining ones gives \( G_2/G_1 = G_3/G_1 = 1/2 \), see Fig. 5; with \( \rho_1 = \rho \) this results in \( c_1 = 1 \) and \( c_2 = c_3 = 1/\sqrt{2} \). One should note that both the strains and velocities for the nonhomogeneous case diminish much faster than for the uniform, homogeneous case. Similar patterns may be observed for other cross-sections and floors as well for the case when \( c_2 = c_3 > 1 \).

The transmitted and reflected shear waves in the columns of the second and third floors travel with somewhat lower/higher speed than on the first floor of this nonhomogeneous structure thus creating decreases of shear strains and velocities at the cross-section shown depending on overall geometry and nonhomogeneity. Comparisons of the results for the base nonhomogeneous structure, with the response spectra of a homogeneous one, show that the isolation through nonhomogeneity (isolation from the external loading at the base) is in general effective in reducing the peak loads transmitted to the superstructure and the strains (stresses) and velocities are significantly lower, compare [11].

Analysis of shear strains and velocities at the cross-sections implies that detailed calculations should be performed when physical properties are changing from element to element (jump-like change in the wave speed from floor to floor). However, since the methods allow for the results to be obtained for any location at any time, it is therefore feasible to simulate the dynamical behaviour of an entire nonhomogeneous structure and choose such physical properties that satisfy overall design criteria.

REFERENCES

Fig. 4 Model of a multi-storey structure

Fig. 5 Equivalent shear modulus versus porosity coefficient

Fig. 6 Shear strain $\frac{\partial y}{\partial x}$ versus time $t$ for the cross-section $x = 0.75$

Fig. 7 Shear strain $\frac{\partial y}{\partial x}$ versus time $t$ for the cross-section $x = 2.25$

Fig. 8 Velocity $\frac{\partial y}{\partial t}$ versus time $t$ for the cross-section $x = 2.25$