FEM Based Modal Analysis of a Damaged Free-Free Beam

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ABSTRACT Dynamic properties are governed by mass, stiffness, and damping. Defects occurring in structural elements will influence their dynamic characteristics because of local stiffness variations. The dynamic behavior of a multiple damaged beam is very complex due to the nonlinearity of the opening and closing of notches or cracks. In this paper, the modal analysis for a three dimensional free-free ceramic beam with five cracks is performed. Mode shapes and natural frequencies are first obtained using Finite Element Model (FEM), based on which a proper state space model is built to describe the dynamic characteristics of the beam including mode response, frequency response, and displacement response. As a representative model of a damaged structure, these can provide an insight into the extent of defects. In addition, a model reduction technique is applied based on sorting the dc gain of the damaged beam.

INTRODUCTION

Defects present in vibrating components could affect their vibration response and finally lead to catastrophic failures. It is necessary to investigate the dynamic behaviors of damaged structures. Modal analysis is an important vibration analysis tool for structural diagnostics during the monitoring and servicing of a structural system. Considerable efforts have been devoted to understand the dynamics of damaged beams through modal analysis [1-10]. Ostachowicz and Krawczuk [6] used a spring to represent the crack section and performed modal analysis for each segment of the beam using appropriate matching conditions at the location of the spring. The stress intensity factor was used to calculate the equivalent stiffness of the spring at the crack location. Chati and Rand [7] addressed the problem of nonlinear dynamics of the infinite degree of freedom cracked beam via calculating piecewise mode shapes and bilinear frequencies. Shaw and Pierre [8, 9] developed a method that is based on invariant manifolds in the state space of nonlinear systems to obtain reduced-order models via nonlinear normal modes. Michael Hatch [10] introduced state space modeling method to simulate the structure vibration.

In this paper, the modal analysis for a damaged free-free ceramic beam is carried out. There are a total of five cracks with different depths at different locations of the beam. The natural frequencies and mode shapes are obtained using a Finite Element Model (FEM). Through the state space model analysis, the dynamic response of the cracked beam including displacement, frequency, and mode responses, and shifts of the frequency are obtained. DC gain and Modred method are introduced for sorting modes and model order reduction.

FINITE ELEMENT MODELING

Mathematically, structural modal analysis is an eigenproblem. The governing equation that governs transverse vibration of continuous Euler beam in bending within the x-y plane under its own weight is given as:

$$\rho A \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) = f(x,t) \tag{1}$$

where ρ is mass density, A is area of cross section, E is Young's modulus of the beam, and I is the area moment of inertia of the beam cross section. Writing the solution in the following form:

$$y(x,t) = Y(x)e^{-ivt} \quad (2)$$

and substituting it into Equation (1), the following eigenvalue problem is obtained:

$$\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 Y}{\partial x^2}) - \rho A \omega^2 Y = 0 \quad (3)$$

The above eigenvalue problem in differential form can be converted to finite element formulation as:

$$-\omega_i^2[M]\{u\} + [K]\{u\} = 0 \quad (4)$$

where [M] and [K] are mass and stiffness matrices given as [11]:

$$[M] = \rho A\int_0^l \psi_i \psi_j dx = \frac{\rho A l}{420}\begin{bmatrix} 156 & -22l & 54 & 13l \\ -22l & 4l^2 & -13l & -3l^2 \\ 54 & -13l & 156 & 22l \\ 13l & -3l^2 & 22l & 4l^2 \end{bmatrix}, i = 0, 1, 2 \quad (5)$$

$$[K] = EI\int_0^l \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}dx = \frac{2EI}{l^3}\begin{bmatrix} 6 & -3l & -6 & -3l \\ -3l & 2l^2 & 3l & l^2 \\ -6 & 3l & 6 & 3l \\ -3l & l^2 & 3l & 2l^2 \end{bmatrix}, i = 0, 1, 2 \quad (6)$$

where ψ is the shape function of the i^{th} degree of freedom, l is the length of the finite element, and ω_i^2 is the corresponding eigenvalue of the beam. The square roots of ω_i^2, $\omega_1, \omega_2 \ldots \omega_n$ are called the natural frequencies of the beam. In addition, there is also an n-dimensional vector called eigenvector U_i, which can be obtained by using Equation (7):

$$\{[K] - \omega_i^2[M]\}U_i = 0 \quad (7)$$

The beam is analyzed for multiple levels of damage. A 3-D free-free ceramic beam with an increasing number of cracks at different locations with different depths was studied. After analyzing the beam with one crack, the beam is considered to have two cracks. After analyzing the beam with two cracks, let the beam have three cracks. Finally there are a total of five cracks on the beam. The crack depths and locations (distance to the left end of the beam) are listed in table 1. The material and geometric parameters used to carry out the analysis are: E = 4.2 x 10^{11} $\mathrm{P_a}$, $\rho = 3.2\ \mathrm{g/cm^3}$, v = 0.21, length (L) = 45 mm, width (b) = 4 mm, and height (h)= 3 mm.

Table 1: Crack depths and locations

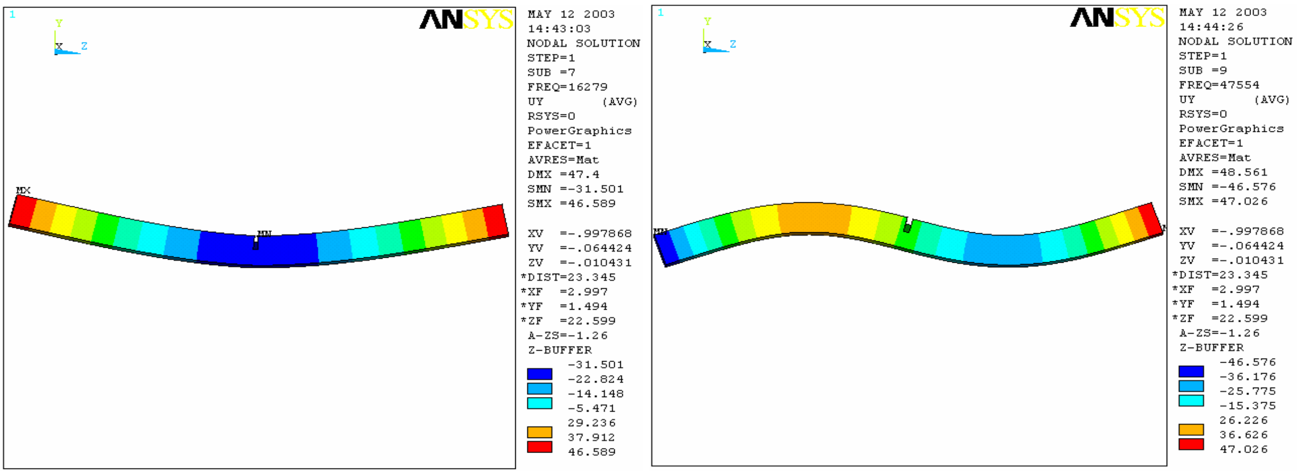
	Crack #1	Crack # 2	Crack # 3	Crack # 4	Crack # 5
Depth (mm)	1.8	0.3	0.8	0.4	0.6
Location (mm)	22.0	30.1	24.9	27.2	29.0

The commercially available Finite Element package ANSYS was used to obtain the numerical results. The beam was modeled using solid, brick, 8-node elements. This kind of element is used for the three-dimensional modeling of solid structures. Each of the eight nodes has the following three degrees of freedom: translations in the nodal x, y, and z directions. To achieve more accurate results, a finer mesh was used around specified cracked areas. Table 2 shows the mesh quality and convergence of the Finite Element mesh refining process for the case of one crack.

The first two transverse mode shapes for the beam with one crack are shown in Figure 1. Since the beam is free-free, in ANSYS, it will give the motion of the beam in six degrees of freedom (UX, UY, UZ, ROTX, ROTY, ROTZ). The FEM results of the full natural frequencies (transversal, longitudinal, and rotational) for the beam with different cracks are shown in Table 3, which shows clearly that the natural frequencies of all the modes decrease with the increase of the number of cracks in the beam.

Table 2: A convergence study for natural frequencies of the beam with one crack

Mode	Frequency (Hz)			
	10,350 Elements	14,080 Elements	24,300 Elements	27,680 Elements
1	17,331	17,080	16,283	16,279
2	22,890	22,580	22,309	22,305
3	48,388	48,008	47,558	47,554



(a) First mode (f = 16,279 Hz), (b) Second mode (f = 47,554 Hz)

Figure 1: Mode shapes for the beam with one crack

Table 3: Comparison for full degrees of freedom natural frequencies

Crack Level	Frequency (Hz)					
	First Mode	Second Mode	Third Mode	Fourth Mode	Fifth Mode	Sixth Mode
Beam without cracks	17,411	22,955	48,398	62,617	75,671	96,330
Beam with one crack	16,279	22,305	47,554	60,411	73,941	85,339
Beam with two cracks	16,270	22,297	47,512	60,394	73,902	85,295
Beam with three cracks	16,173	22,232	47,454	60,331	73,660	85,176
Beam with four cracks	16,115	22,189	47,408	60,262	73,543	85,097
Beam with five cracks	16,029	22,132	46,961	60,056	72,706	84,586

The free-free beam has a large number of vibration modes, which require a large amount of computation time. Since most of the higher modes and some of the degrees of freedom have lower analysis value, there is a need to build an appropriate state space frequency response model adjunct to the FEM model analysis to separate and examine the important transverse vibration responses of the beam. This state space model can describe dynamic characteristics of the beam, including dc gain value, frequency response, and displacement response.

STATE SPACE MODEL ANALYSIS

The damped equation of motion is described as:

$$\ddot{x}_{li} + 2\zeta w_i \dot{x}_{li} + w_i^2 x_{li} = F_{li} \tag{8}$$

where ζ is the critical damping, w_i is the i^{th} eigenvalue, and F_{li} is the applied force. Assuming that only three modes are considered, the corresponding equations of motion are:

$$\ddot{x}_{l1} + 2\zeta w_1 \dot{x}_{l1} + w_1^2 x_{l1} = F_{l1} \tag{9}$$

$$\ddot{x}_{l2} + 2\zeta w_2 \dot{x}_{l2} + w_2^2 x_{l2} = F_{l2} \tag{10}$$

$$\ddot{x}_{l3} + 2\zeta w_3 \dot{x}_{l3} + w_3^2 x_{l3} = F_{l3} \tag{11}$$

In state space matrix form, $\dot{x} = Ax + Bu$, Equations (9) - (11) reduce to the following state space equations of motion:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = 0 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -w_1^2 & -2\zeta_1 w_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -w_2^2 & -2\zeta_2 w_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -w_3^2 & -2\zeta_3 w_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ F_{l1} \\ 0 \\ F_{l2} \\ 0 \\ F_{l3} \end{bmatrix} u \tag{12}$$

Equation (12) can be solved to get the frequency and time domain response. The matrix A is made up of eigenvalue and the damping for each mode. Matrix B is made up of the applied force at the node. The state space modal form may then be rewritten for each mode as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_1^2 & -2\zeta_1 w_1 \end{bmatrix} + \begin{bmatrix} 0 \\ F_{l1} \end{bmatrix} u \quad \text{mode 1} \tag{13}$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_2^2 & -2\zeta_2 w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_{l2} \end{bmatrix} u \quad \text{mode 2} \tag{14}$$

$$\begin{bmatrix} \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_2^2 & -2\zeta_2 w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_{l3} \end{bmatrix} u \quad \text{mode 3} \tag{15}$$

To identify and sort which modes are critical to the overall beam vibration, the dc gain value in the y-direction (transverse) is used. For both the damped and undamped system, a transfer function is defined as [10]:

$$\frac{V_j}{F_m} = \sum_{i=1}^{k} \frac{V_{nji} V_{nmi}}{s^2 + \omega_i^2} \tag{16}$$

$$\frac{V_j}{F_m} = \sum_{i=1}^{k} \frac{V_{nji} V_{nmi}}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \tag{17}$$

where $V_{nji} V_{nmi}$ is the product of the j^{th} row of the output eigenvector and the m^{th} row of the i^{th} input, and ω_i is the resonant frequency which is the eigenvalue of the i^{th} mode. Letting $s = j\omega = 0$, we get the i^{th} mode frequency response at dc. Therefore, dc gain is defined as:

$$\frac{V_{ji}}{F_{mi}} = \frac{V_{nji}V_{nmi}}{\omega_i^2} \tag{18}$$

For Equation (17), letting $s = j\omega$ and $s^2 = -\omega_i^2$, the peak gain amplitude of each resonance mode is then given as:

$$\text{peak gain } = \frac{-j}{2\zeta_i}\left(\frac{V_{nji}V_{nmi}}{\omega_i^2}\right) \tag{19}$$

Therefore, we have:

$$\text{peak gain} = \frac{-j}{2\zeta_i} \times \text{dc gain} \tag{20}$$

The relationship between the dc gain and peak gain for a mode is that the dc gain term is divided by $2\zeta_i$ and multiplied by -j, which gives a -90^0 phase shift at resonance. ζ_i is the critical damping which is very small and hence amplifies the response with a resonant peak.

Typically, only the dynamic responses of modes that exhibit large amplitudes and/or lie in a certain frequency range are of interest. To obtain an accurate model of a system of coupled differential equations, the modes that lie outside the frequency range of interest are first truncated, and then the modes that have small amplitudes are also truncated. The contribution of the truncated modes can be described by the direct dynamic simulation of these modes. An algorithm named Modred method [10] can be used to truncate the modes that are not of interest. The state space equations are given as:

$$\dot{x} = Ax + Bu \tag{21}$$

$$y = Cx + Du \tag{22}$$

To separate the most important modes from the less important modes, the matrix A, B, and C are rearranged. The state vector is partitioned into x_k (to be kept) and x_e (to be eliminated):

$$\begin{bmatrix} \dot{x}_k \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_{kk} & A_{ke} \\ A_{ek} & A_{ee} \end{bmatrix} \begin{bmatrix} x_k \\ x_e \end{bmatrix} + \begin{bmatrix} B_k \\ B_e \end{bmatrix} u \tag{23}$$

$$y = \begin{bmatrix} C_k & C_e \end{bmatrix} \begin{bmatrix} x_k \\ x_e \end{bmatrix} + Du \tag{24}$$

Setting $\dot{x}_e$=0 and solving for x_e leads to:

$$x_e = -A_{ee}^{-1}A_{ek}x_k - A_{ee}^{-1}B_e u \tag{25}$$

Substituting into Equation 23 yields:

$$\dot{x}_k = (A_{kk} - A_{ke}A_{ee}^{-1}A_{ek})x_k + (B_k - A_{ke}A_{ee}^{-1}B_e)u \tag{26}$$

The output y can then be expressed as:

$$y = (C_k - C_e A_{ee}^{-1}A_{ek})x_k + (D - C_e A_{ee}^{-1}B_e)u \tag{27}$$

And the new state equations are:

$$\dot{x}_r = A_r x_r + B_r u \tag{28}$$

$$y_r = C_r x_r + D_r u \tag{29}$$

where:

$$A_r = A_{kk} - A_{ke} A_{ee}^{-1} A_{ek} \quad (30)$$

$$B_r = B_k - A_{ke} A_{ee}^{-1} B_e \quad (31)$$

$$C_r = C_k - C_e A_{ee}^{-1} A_{ek} \quad (32)$$

$$D_r = D - C_e A_{ee}^{-1} B_e \quad (33)$$

This technique can be used to truncate the lower dc gain modes, while retaining the overall system dc gain.

RESULTS AND DISCUSSION

Figure 2 shows the plot of frequency versus mode number for the beam with one crack, which helps to understand the resonant frequencies of the damaged beam in a particular range of interest.

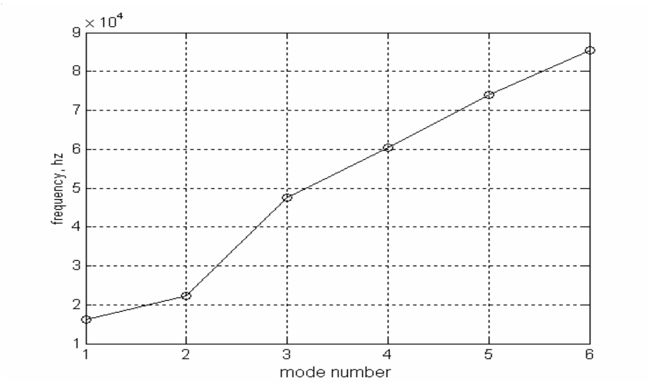


Figure 2: Frequency versus mode number for the beam with one crack

Figure 3 shows the dc gain value of each mode (total 6 modes included) versus the mode number for the beam with one crack. The modes are sorted by dc gain in the y-displacement component (transverse vibration). Mode 2 and mode 4 are vibration modes in the x-displacement (longitudinal vibration) and have small dc gain values in the y-direction, as shown in Figure 3. That means mode 2 and mode 4 are less important and have less contribution to the total transverse vibration of the beam than mode 1, 3, 5, and 6. Because the rotational vibration also has displacements (eigenvectors) in the y-direction, it is possible that the fifth mode (the first mode of rotational vibration) has higher dc value than the sixth mode (the third transverse vibration in the y-direction).

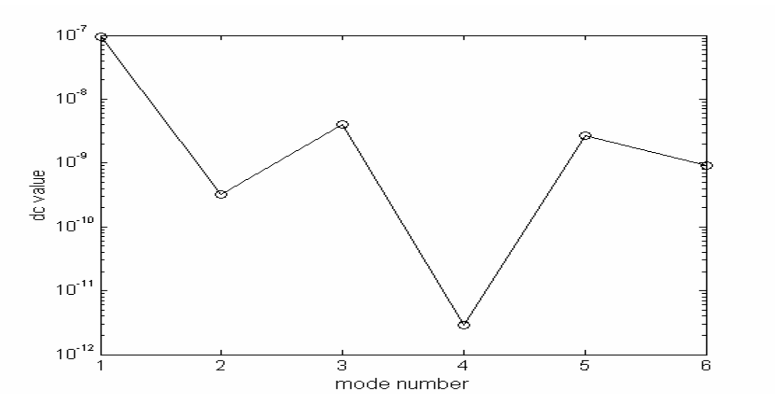


Figure 3: DC values versus mode number for the beam with one crack

The occurrence of well separated vibration modes allows the identification of the modal parameters to be done according to the state space and Modred order reduction technique. Figure 4 shows the overall frequency response with overlaid individual mode contributions for all the 6 modes for the beam with one crack.

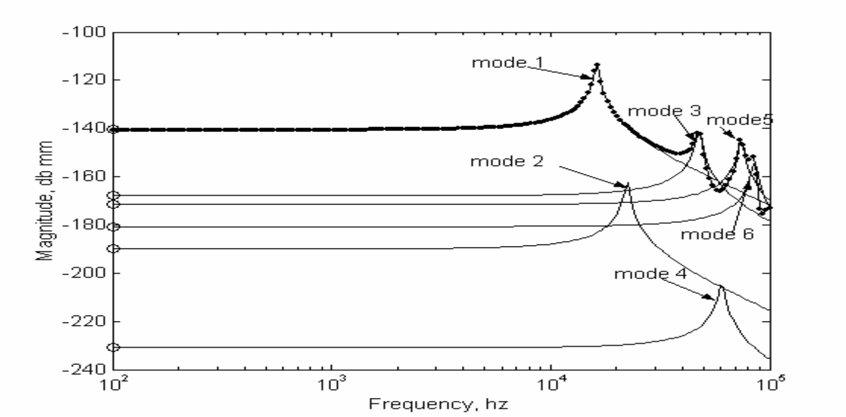


Figure 4: The overall frequency response with 6 modes for the beam with one crack

Comparing the change in natural frequencies, Figure 5 shows the shift of the fundamental natural frequency (first mode) for the beam without crack and for the beam with one crack.

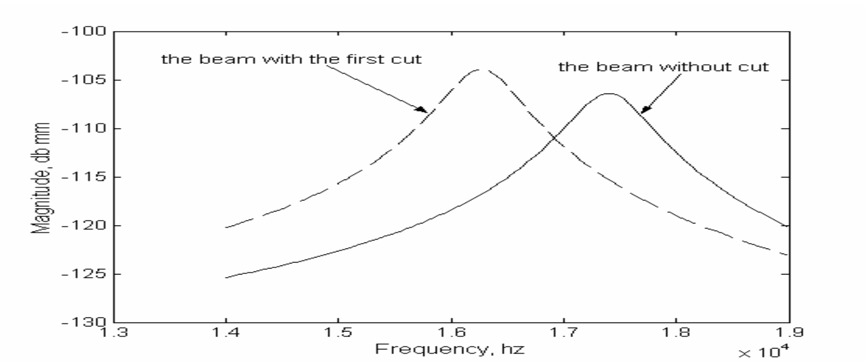


Figure 5: The shift of the fundamental (first mode) natural frequency

Now consider the free-free beam with a total of five cracks. The final frequency response is shown in Figure 6. It indicates that with the increase of the number of cracks, the vibration response is still similar to that of the beam with only one crack. Mode 2 and 4 still have small dc gain values and have less contribution to the overall vibration.

Figure 7 shows the natural frequencies of the first mode for the beam without crack (A), one crack (B), two cracks (C), three cracks (D), four cracks (E), and five cracks (F). It can be seen that the natural frequency exhibits strong shift due to the presence of cracks. Comparing case B (one crack) with case C (two cracks), it appears that the second crack has no significant influence on the frequency shift due to its small size. Similarly, comparing C with D, D with E, and E with F, Figure 7 shows that bigger crack results in larger natural frequency shift.

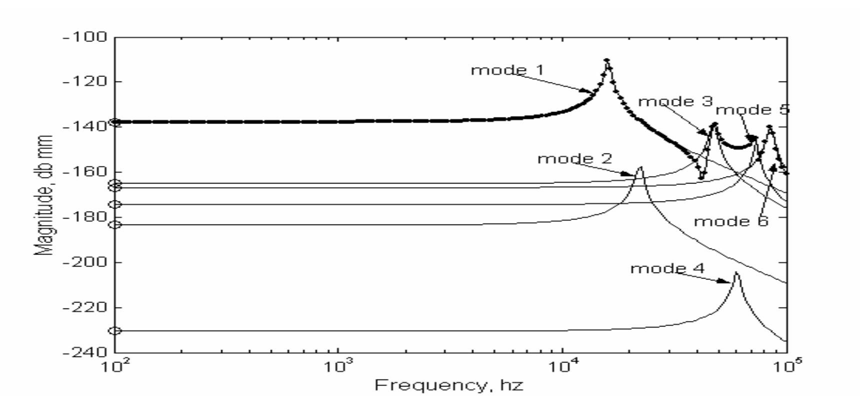


Figure 6: Frequency response for the beam with five cracks, 6 modes included

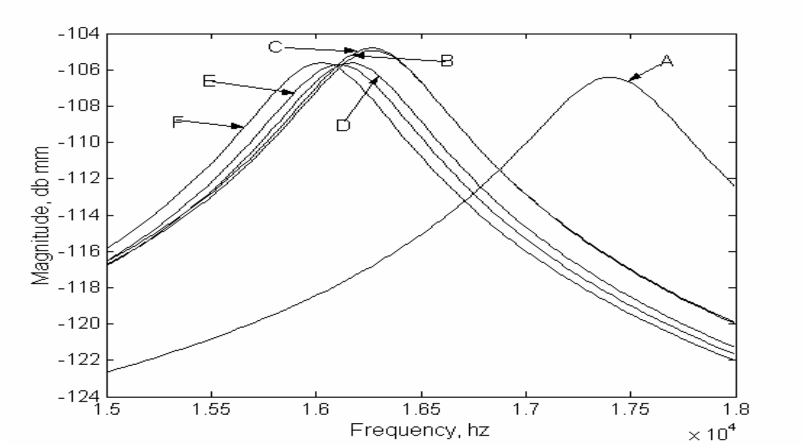


Figure 7: Shifts of the natural frequency (first mode)
A— the beam without cracks; B— the beam with the one crack; C— the beam with the two cracks;
D— the beam with three cracks; E— the beam with four cracks; F— the beam with five cracks

CONCLUSIONS

The main goal of this paper is to understand dynamic responses of a cracked beam using FEM based modal analysis. A ceramic beam with a total of five cracks has been studied. The numerical results were obtained by modeling the cracked beam with solid brick 8-node elements. A proper state space model was built using the eigenvalues and eigenvectors from the FEM model. After sorting the modes using dc gain values, an algorithm named Modred method was used to truncate the modes that have less contribution to the total vibration response, so that only the important modes within the range of interest are extracted and analyzed. It shows that the modes response have no significant changes compared with that of the beam without cracks. However, the shift of natural frequency due to the crack is obvious, which increases with the increasing crack number and shows to be sensitive to the crack size.

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